Samuelson's Correspondence Principle Reassessed

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Abstract

A theorem of Krasnosel'ski and Zabreiko (1984) implies that an equilibrium of an abstract economic model cannot be asymptotically stable, for natural adjustment dynamics, unless its fixed point index is +1. This result provides a precise and general formulation, and proof, of Samuelson's correspondence principle, which is commonly understood as a consequence of the 1-dimensional case.

Keywords: Paul Samuelson, correspondence principle, fixed point index, vector field index, dynamical system, asymptotic stability, adjustment to equilibrium, Lyapunov function, converse Lyapunov theorem.

1 Introduction

The correspondence principle was described by Paul Samuelson in two articles (Samuelson (1941, 1942)) and his famous *Foundations of Economic Analysis* (Samuelson (1947)) after being stated informally by Hicks (1939). Roughly, it asserts that the stability of an equilibrium, with respect to dynamic equilibration processes, implies that the equilibrium's comparative statics have certain qualitative properties. Samuelson's writings consider a host of specific models, but he did not formulate the correspondence principle as a definite and general theorem. This note points out that such a formulation is possible, and in fact already exists. Specifically, a theorem of Krasnosel'ski and Zabreiko (1984) in the theory of nonlinear dynamical systems gives a necessary condition for stability of an isolated equilibrium in the context of a finite dimensional dynamical system. (Theorem 2 of Dierker (1972) is closely related, as we explain in Section 2.) The correspondence

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principle, as it is commonly understood, follows directly from the 1-dimensional case of this result, and the consequences of stability for comparative statics in higher dimensions are immediately evident.

Since Samuelson's work a vast body of research studying dynamic economic models has developed, but for the most part it studies equilibria that play out over time, rather than dynamic adjustment to equilibrium. Researchers in general equilibrium theory (e.g., Arrow and Hurwicz (1958); Arrow et al. (1959)) found some special cases in which some equilibria are necessarily stable with respect to Walrasian tatonnement dynamics, but examples developed by Scarf (1960) showed that this phenomenon is restricted to very small numbers of goods or agents. A later line of research (Saari and Simon (1978); Saari (1985); Williams (1985); Jordan (1987)) showed that stability is informationally demanding, in the sense that an adjustment process that is guaranteed to return to equilibrium after a small perturbation requires essentially all the information in the matrix of partial derivatives of the aggregate excess demand function. It should be stressed that these findings have limited relevance here, because in the correspondence principle stability is an hypothesis rather than a conclusion. Even though the correspondence principle is now venerable and very widely known, research specifically related to it has been rather sparse and largely restricted to 1-dimensional applications, with the notable exception of its development by Echenique (2000, 2002, 2004) in the context of games with strategic complementarities. Echenique works with discrete time adjustment processes, and his setting is largely free of topological restrictions. Echenique (2008) provides a succinct guide to the current state of our understanding and related literature.

The difficulties impeding a clearer understanding of the correspondence principle are both conceptual and mathematical. On the conceptual side, Samuelson's understanding of the role of dynamics in economics was based on physics, and would nowadays be considered rather naive. Specifically, he imagined that equilibrium was an endstate of a dynamic process that was potentially an object of theoretical and empirical investigation, and in practice the correspondence principle is derived from Walrasian tatonnement. After the rational expectations revolution this perspective seems problematic: if a continuous adjustment process leads to equilibrium, and the agents in the model understand this, instead of conforming to the process they will exploit it. The nature of the equilibration process is therefore unknowable in principle, or perhaps not even a meaningful concept. In the face of these concerns, what might we mean when we say that an equilibrium is dynamically stable?

In evolutionary game theory the principle of rational expectations does not apply, because the dynamics are given by reproduction rates that depend only on the success of each strategy in the current environment. Thus it is natural that an important generalization of the result emphasized here, due to Demichelis and Ritzberger (2003), emerged in that literature. Their paper provides pointers to a rather large literature in which evolutionary processes are studied, with various consequences for Nash equilibria and sets of equilibria that are, in various senses, stable.

A quite different conceptual picture emerges when we consider how the correspondence principle is actually applied. Both in general equilibrium theory and in game theory, as well as in many other economic models, equilibrium is defined, in at least an informal sense, as a rest point of a process in which utility maximizing agents would respond to a failure of the equilibrium conditions by changing their behavior. This gives rise to some sense of "reasonable" or "natural" dynamics: the various prices each adjust in the direction of excess demand, though possibly at quite different rates, or each agent adjusts her mixed strategy in some direction that would increase expected utility if other agents were not also adjusting. Our main result can be understood as asserting that if an equilibrium is stable for a dynamical system that is reasonable or natural in this sense, then a certain linear transformation (roughly, the one represented by the matrix that is inverted in the process of computing comparative statics) has a positive determinant. If we think of this picture as a "justification" of the equilibrium, in the sense of asserting that because the determinant is positive, the equilibrium is in some sense likely, then the story seems rather weak. But this is not really the point.

Consider an equilibrium for which the determinant is negative. There is *no* reasonable or natural dynamics for which the equilibrium is stable. From a strictly logical point of view it is still possible that the equilibrium is stable, and thus potentially relevant empirically, in relation to some actual adjustment process, which must be either unnatural, or perhaps much more complex in some mysterious and unfathomable way. It seem likely that most economists would regard this possibility as far fetched. This creates a strong presumption that the correspondence principle is a valid criterion for ruling some equilibria out of consideration, and indeed should be an element of the fundamental toolkit of economic analysis.

The next section states the Krasnosel'ski-Zabreiko theorem, and explains its consequences for comparative statics. Section 3 gives an informal discussion of the elements of one method of proving this result; the topological fixed point index is a central aspect of this approach. Section 4 concludes with a few brief remarks.

2 The Main Result

Let $U \subset \mathbb{R}^m$ be open. Elements of U are thought of as vectors of endogenous variables. The economic model is given by a function $g : U \to \mathbb{R}^m$. We will think of g as a vector field, and in general we will describe a function from a subset of U to $\mathbb{R}^m$ as a vector field when we wish to highlight this interpretation. Under mild technical conditions described below such a vector field determines a dynamical system. Let $f : U \to \mathbb{R}^m$ be the function $f(x) = x + g(x)$. We think of f as a function whose fixed points are of interest. An *equilibrium* is an $x_0 \in U$ such that $g(x_0) = 0$, so x_0 is a rest point of the dynamical system determined by g. Equivalently, an equilibrium is a fixed point of f. When f and g are C^1 the *index transformation* at x_0 is the linear transformation

$$-Dg(x_0) = I - Df(x_0) : \mathbb{R}^m \to \mathbb{R}^m$$

where I is the identity. An equilibrium is *regular* if its index transformation is nonsingular.

We now review basic definitions and results concerning dynamical systems. Recall that if (Y, d) and (Z, e) are metric spaces, a function $\gamma : Y \to Z$ is *locally Lipschitz* if each $y \in Y$ has a neighborhood V_y for which there is a constant $L_y > 0$ such that $e(\gamma(y'), \gamma(y'')) \le L_y d(y', y'')$ for all $y', y'' \in V_y$. The Picard-Lindelöf theorem asserts that if $C \subset U$ is compact and g is locally Lipschitz, as we assume henceforth, then for some $\varepsilon > 0$ there is a unique continuous function $\varphi : C \times (-\varepsilon, \varepsilon) \to U$ such that $\varphi(x, 0) = 0$ for all x, each $\varphi(x, \cdot)$ is C^1, and $\frac{\partial \varphi}{\partial t}(x, t) = g(\varphi(x, t))$ for all (x, t). In addition, if g is C^r for some $1 \le r \le \infty$, then φ is C^r. Using this result, it is not hard to show that there is an open $\Omega' \subset U \times \mathbb{R}$ and a continuous function $\Phi' : \Omega' \to U$ such that

(a) for each $x \in U$, $\{\, t : (x, t) \in \Omega' \,\}$ is an open interval containing 0;

(b) $\frac{\partial \Phi'}{\partial t}(x, t) = g(\Phi'(x, t))$ for all $(x, t) \in \Omega'$.

If Ω'' and Φ'' also satisfy these conditions, then $\Omega' \cup \Omega''$ satisfies (a), and the uniqueness aspect of the Picard-Lindelöf theorem implies that Φ' and Φ'' agree on $\Omega' \cap \Omega''$, so the function on $\Omega' \cup \Omega''$ that agrees with Φ' on Ω' and with Φ'' on Ω'' is well defined and satisfies (b). It follows that there is a pair Ω and Φ satisfying these conditions that is maximal in the sense that $\Omega' \subset \Omega$ for any other such pair Ω' and Φ'. We call Ω and Φ the *flow domain* and the *flow* of the dynamical system given by g.

The fundamental definitions related to stability are as follows. A set $A \subset U$ is *stable* if, for every neighborhood V of A, there is a neighborhood W such that $W \times [0, \infty) \subset \Omega$

and $\Phi(x,t) \in V$ for all $x \in W$ and $t \geq 0$. Stability does not require the convergence to A of trajectories that start near A.

Asymptotic stability is a stronger concept. The *ω-limit set* of $x \in U$ is

$$\bigcap_{t_0 \geq 0} \overline{\{\, \Phi(x,t) : t \geq t_0 \,\}}.$$

A set $A \subset U$ is *invariant* if $A \times [0,\infty) \subset \Omega$ and $\Phi(A,t) \subset A$ for all $t \geq 0$. The *domain of attraction* of A is

$$D(A) = \{\, x \in U : \text{the } \omega\text{-limit set of } x \text{ is nonempty and contained in } A \,\}.$$

A set $A \subset U$ is *asymptotically stable* if:

(a) A is compact;

(b) A is invariant;

(c) $D(A)$ is a neighborhood of A;

(d) for every neighborhood $\tilde{V}$ of A there is a neighborhood V such that $\Phi(x,t) \in \tilde{V}$ for all $x \in V$ and $t \geq 0$.

An asymptotically stable set is *minimal* if it is nonempty and it does not have a nonempty proper subset that is asymptotically stable. If A is asymptotically stable, then so is $\bigcap_{t \geq 0} \Phi(A,t)$, so the two sets are equal when A is minimal. Note that a minimal asymptotically stable set is necessarily connected.

Theorem 1 (Krasnosel'ski and Zabreiko (1984), Th. 52.1)**.** *If g is C^1, x_0 is a regular equilibrium, and $\{x_0\}$ is an asymptotically stable set, then the determinant of the index transformation at x_0 is positive.*

An equilibrium x_0 is *exponentially stable* if there is an invariant neighborhood U of x_0 and constants $B, c > 0$ such that $\|\Phi(x,t) - x_0\| \leq Be^{-ct}\|x - x_0\|$ for all $x \in U$ and $t \geq 0$. If g is C^1, x_0 is an equilibrium, and all the eigenvalues of the $Dg(x_0)$ have negative real parts, then x_0 is exponentially stable. (See p. 181 of Hirsch and Smale (1974). Also, see p. 187 for the fact that if $\{x_0\}$ is stable, then all the eigenvalues have nonpositive real parts.) Theorem 2 of Dierker (1972) is based on the observation that if all the the eigenvalues of $Dg(x_0)$ have negative real parts, then the vector field index (this and the fixed point index are described in the next section) of the equilibrium is $(-1)^m$, which implies that its fixed point index is $+1$ and the determinant of the index transformation is positive. In the general equilibrium context of that paper the sum of the fixed point indices is $+1$, so it cannot be the case that all equilibria satisfy this condition unless there is a unique equilibrium. For additional details see Ch. 11 of Dierker (1974).

In order to explain the relevance of Theorem 1 to comparative statics we consider a parameterized model $g : U \times P \to \mathbb{R}^m$ where $P \subset \mathbb{R}^p$ is an open set of vectors of exogenous parameters. The method of comparative statics is to totally differentiate the equation $g(x(\alpha), \alpha) = 0$, then rearrange, arriving at the equation

$$\frac{dx}{d\alpha}(\alpha_0) = -\partial_x g(x_0, \alpha_0)^{-1} \partial_\alpha g(x_0, \alpha_0).$$

Here $\partial_x g$ and $\partial_\alpha g$ denote the matrices of partial derivatives of the components of g with respect to the components of x and α respectively. This calculation is justified when g is C^1 and x_0 is a regular equilibrium for α_0 because the implicit function theorem implies that there is a C^1 function x defined in a neighborhood of α_0 such that $x(\alpha_0) = x_0$ and $g(x(\alpha), \alpha) = 0$ for all α in the domain of x.

When $m = 1$, $-\partial_x g(x_0, \alpha_0)$ and its inverse are just numbers, and if they are positive, then $\frac{dx}{d\alpha}(\alpha_0)$ is a positive scalar multiple of $\partial_\alpha g(x_0, \alpha_0)$. This phenomenon is what is commonly understood as the correspondence principle. It is sometimes asserted (e.g., the critical discussion on pp. 320–321 of Arrow and Hahn (1971)) that the correspondence principle has no consequences in higher dimensional settings. This does not seem quite correct. The fact that the determinant of $-\partial_x g(x_0, \alpha_0)$ is positive is just one bit of information, and its consequences are not as simple as in the one dimensional case, but it is nonetheless a qualitative feature of the comparative statics. One should expect that it will find some interesting multidimensional applications, in ways that are perhaps difficult to foresee, because they will depend on auxilary assumptions motivated by specific contexts.

3 Mathematical Background

There is a great deal to be learned from Krasnosel'ski and Zabreiko (1984), but it does not provide a succinct self contained proof of Theorem 1, because the argument is wrapped up in larger themes developed throughout the course of the book. Zabczyk (1992), pp. 109–111, gives a proof that is brief and fairly simple. Also, the proofs of the more general result of Demichelis and Ritzberger (2003) are useful. Here we attempt to give an informal and intuitive explanation of the main elements of the relevant circle of ideas, along the lines of Demichelis and Ritzberger. The first chapter of McLennan (2012) provides a more extended but still informal discussion. The rest of that book is a comprehensive development of fixed point theory, emphasizing the fixed point index, which has a central role in what follows.

The first step in taking advantage of the hypothesis of stability is to attain an invariant neighborhood of x_0. The tool used to achieve this is a Lyapunov function. A function $h : U \to \mathbb{R}$ is *g-differentiable* if the *g-derivative*

$$gh(x) = \frac{d}{dt} h(\Phi(x,t))|_{t=0}$$

is defined for every $x \in U$. A continuous function $L : U \to [0, \infty)$ is a *Lyapunov function for* $A \subset U$ if:

(a) $L^{-1}(0) = A$;

(b) L is g-differentiable with $gL(x) < 0$ for all $x \in U \setminus A$;

(c) for every neighborhood V of A there is an $\varepsilon > 0$ such that $L^{-1}([0,\varepsilon]) \subset V$.

It is intuitive and very well known that if A is compact and there is a Lyapunov function for A, then A is asymptotically stable. The converse is a highly nontrivial result with a rather complicated history, that is briefly sketched by Nadzieja (1990). Briefly, a sequence of partial solutions, over several decades, eventually culminated in a complete (in the sense that the Lyapunov function can be required to be C^∞) solution by Wilson (1969). We only need the following less refined result, for which Nadzieja's somewhat simpler argument is sufficient.

Proposition 1. *If A is asymptotically stable, then (after replacing U with a suitable neighborhood of A) there is a Lyapunov function for A.*

Let $L : U \to [0, \infty)$ be a Lyapunov function for A. There are some details to attend to in a formal argument, but it should come as no surprise that for some $\varepsilon > 0$, $L^{-1}\big([0,\varepsilon]\big)$ is invariant, with $\bigcap_{t \geq 0} \Phi(L^{-1}([0,\varepsilon]), t) \subset A$.

The fixed point index allows us to take advantage of this information. The theory of the fixed point index begins with the observation that a well behaved function from the unit interval to itself crosses the diagonal going from above to below one more time than it crosses going from below to above. (See Figure 1.) This principle—that the number of fixed points at which the determinant of the index transformation is positive is one greater than the number of fixed points at which the determinant is negative—extends to well behaved functions from the m-dimensional ball to itself, for any m. In the simplest settings one may define the index of a fixed point of the first type to be $+1$ and the index of the second type to be -1. More generally, if $C \subset U$ is compact and $f : C \to U$ has only regular fixed points, none of which lie in ∂C, then the index $\Lambda(f)$ is defined to be the number of fixed points whose index transformations have positive determinants minus the number of fixed points whose index transformations have negative determinants. (If $C \subset U$ is compact we let ∂C denote the topological boundary of C.)

The fixed point index for regular economies was introduced in general equilibrium theory by Dierker (1972, 1974). It plays a role in the analysis of the Lemke-Howson algorithm in Shapley (1974). Hofbauer (1990) applies the vector field index (defined below) to dynamic issues in evolutionary game theory, and Ritzberger (1994) applies it to game theory systematically.

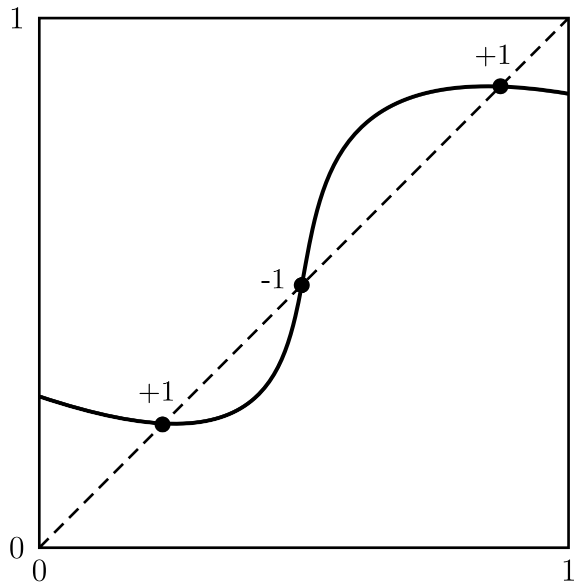

Figure 1

For less well behaved functions the fixed point index is not defined directly, but is instead characterized axiomatically. An *index admissible function* is a continuous function $\varphi : C \to \mathbb{R}^m$, where $C \subset U$ is compact, such that $\varphi(x) \neq x$ for all $x \in \partial C$. Let $\mathcal{I}$ be the set of index admissible functions.

Proposition 2. *There is a unique function $\Lambda : \mathcal{I} \to \mathbb{Z}$, called the* fixed point index*, satisfying:*

(I1) (Normalization) If $c : C \to \mathbb{R}^m$ is a constant function whose value is an element of the interior of C, then $\Lambda(c) = 1$.

(I2) (Additivity) If $\varphi : C \to \mathbb{R}^m$ is an element of $\mathcal{I}$, $C_1, \ldots, C_r$ are pairwise disjoint compact subsets of C, and the fixed points of φ are contained in the union of the

interiors of the C_i*, then*

$$\Lambda(\varphi) = \sum_i \Lambda(\varphi|_{C_i}).$$

(I3) (Continuity) For each element $\varphi : C \to \mathbb{R}^m$ *of* $\mathcal{I}$ *there is a neighborhood* $V \subset C \times \mathbb{R}^m$ *of the graph of* φ *such that* $\Lambda(\tilde{\varphi}) = \Lambda(\varphi)$ *for every* $\tilde{\varphi} \in \mathcal{I}$ *whose graph is contained in* U*.*

Proofs of Proposition 2 are rather lengthy, and depend on background from differential or algebraic topology. In the style of proof employing differential topology the main idea is to approximate an arbitrary continuous function with a smooth function whose fixed points are regular (that is, the index transformation is nonsingular) as shown in Figure 1. This gives a definition of the general index, and of course one needs to show that this definition is well posed, in the sense that all sufficiently accurate approximations give the same definition, and one needs to verify the axioms. Uniqueness is demonstrated by starting with Normalization and using the other axioms to show that a series of increasingly general cases are uniquely determined. In fact Proposition 2 can be generalized in several directions, including much more general spaces, which may be infinite dimensional, and contractible valued correspondences, and the fixed point index has additional properties; informal explanations are given in McLennan (2008) and Chapter 1 of McLennan (2012).

An *index admissible homotopy* is a continuous function $h : C \times [0,1] \to \mathbb{R}^m$ such that for each t, $h_t = h(\cdot,t) \in \mathcal{I}$. In this circumstance we say that h_0 and h_1 are *index admissible homotopic*. Note that Continuity implies that $\Lambda(h_t)$ is a locally constant function of t, hence constant because the unit interval is connected, so $\Lambda(h_0) = \Lambda(h_1)$. For all $0 < t < T < \infty$, $\Phi(\cdot,t)|_{L^{-1}([0,\varepsilon])}$ and $\Phi(\cdot,T)|_{L^{-1}([0,\varepsilon])}$ are index admissible homotopic, and the image of $\Phi(\cdot,T)|_{L^{-1}([0,\varepsilon])}$ is contained in a small neighborhood of A when T is large.

The case of general A is quite interesting, but presents technical complications, so we only state the result achieved by Demichelis and Ritzberger (2003). In the context of a game theoretic model they show that if A is a minimal asymptotically stable subset of the set of equilibria, and A is an absolute neighborhood retract, then

$$\Lambda(\Phi(\cdot,t)|_{L^{-1}([0,\varepsilon])}) = \chi(A)$$

for all $t > 0$, where $\chi(A)$ is the Euler characteristic of A.

From this point forward we assume that $A = \{x_0\}$. For large T the image of $\Phi(\cdot,T)|_{L^{-1}([0,\varepsilon])}$ is contained in a small neighborhood of x_0, in which case convex combination gives an index admissible homotopy between $\Phi(\cdot,T)|_{L^{-1}([0,\varepsilon])}$ and the constant function with value x_0. Applying Normalization and Continuity, we conclude that for all $t > 0$,

$$\Lambda(\Phi(\cdot, t)|_{L^{-1}([0,\varepsilon])}) = 1.$$

There is a variant of the index concept for vector fields that is prominent in the theory of dynamical systems. If $C \subset U$ is compact, a continuous vector field $\gamma : C \to \mathbb{R}^m$ is *index admissible* if it has no equilibria in ∂C. Let $\mathcal{V}$ be the set of index admissible vector fields.

Proposition 3. *There is a unique function* ind : $\mathcal{V} \to \mathbb{Z}$*, called the* vector field index*, such that for all $\gamma \in \mathcal{V}$ with domain C:*

(V1) $\text{ind}(\gamma) = 1$ *if, for some x_0 in the interior of C, $\gamma(x) = x - x_0$ for all $x \in C$.*

(V2) $\text{ind}(\gamma) = \sum_{i=1}^{s} \text{ind}(\gamma|_{C_i})$ *whenever $C_1, \ldots, C_s$ are pairwise disjoint compact subsets of C such that all the equilibria of γ are contained in the union of the interiors of the C_i.*

(V3) There is a neighborhood $V \subset C \times \mathbb{R}^m$ of the graph of γ such that $\text{ind}(\gamma') = \text{ind}(\gamma)$ *for any vector field γ' on C whose graph is contained in V.*

A simple method of proving Proposition 3 connects the vector field index to the fixed point index. For any $\gamma \in \mathcal{V}$ and any neighborhood $V \subset C \times \mathbb{R}^m$ of the graph of γ there is C^∞ vector field $\tilde{\gamma}$ on C whose graph is contained in V. If V is sufficiently small, then $\tilde{\gamma}$ is necessarily index admissible. There is a C^∞ extension of $\tilde{\gamma}$ to a neighborhood of C (by virtue of the definition of a C^∞ function on an arbitrary subset of U) which defines a dynamical system with flow Ψ, and we can set

$$\text{ind}(\gamma) = \Lambda(\Psi(\cdot, t)|_C)$$

for small *negative* t. One must show that this definition does not depend on the choice of $\tilde{\gamma}$ if V is sufficiently small, or on the choice of the extension. One must also show that there is some $\delta > 0$ such that $\Psi(\cdot, t)|_C$ is index admissible whenever $-\delta < t < 0$, so that the choice of t does not matter. And of course one must verify the axioms. All this is possible; cf. Ch. 15 of McLennan (2012).

An obvious and basic principle of dynamical systems is that replacing a vector field with its negation amounts to reversing the direction of time. We conclude that for all $t > 0$,

$$1 = \Lambda(\Phi(\cdot, t)|_{L^{-1}([0,\varepsilon])}) = \text{ind}(-g|_{L^{-1}([0,\varepsilon])}).$$

Note that up to this point the argument has not used the assumption that x_0 is a regular equilibrium, and in fact *the real assertion of Theorem 1, that the index of* $\{x_0\}$ *for* $-g$ *is* $+1$*, remains true without this assumption.* The axiomatic description of the index (which is not standard in the theory of dynamical systems) is required in order to be able to *state* this version of the result.

If x_0 is a regular equilibrium, then in a small neighborhood of x_0 one may easily construct an index admissible (in the sense that is appropriate for vector fields) homotopy between g and the linear vector field $x \mapsto Dg(x_0)(x - x_0)$. The set of nonsingular linear transformations from $\mathbb{R}^m$ to itself has two path components, according to the sign of the determinant, so (V1) and (V3) imply that the determinant of $-Dg(x_0)$ is positive, which is the assertion of Theorem 1.

4 Concluding Remarks

Samuelson's correspondence principle has been revealed to be the consequence, for comparative statics, of a prominent theorem from the theory of dynamical systems. Seen in this light, there is a compelling argument for regarding it as one of the foundational principles of economic analysis, with potentially interesting consequences beyond the 1-dimensional case. Our proof sketch reveals the central role and utility of the fixed point index. As additional evidence of the value of the fixed point index in economic analysis we mention the proof of uniqueness of equilibrium expected payoffs for a coalitional bargaining game in Eraslan and McLennan (2005).

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