A Theoretical Foundation for the Undercut-Proof Equilibrium*

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Abstract

This paper develops a theoretical foundation for the undercut-proof equilibrium (see Shy, 1996, 2002; Morgan and Shy, 2013). In a general spatial setting, the set of undercut-proof prices is equivalent the core of a non-transferable utility cooperative-game, played on the set of outcomes that are feasible in Bertrand competition. The result depends critically on two conditions: First, firms must have unlimited capacity and constant marginal costs. Second, the goods produced by firms must only be differentiated by the spatial characteristics of the market. Examples on the Hotelling line and network markets show how the undercut-proof equilibrium can be used to describe stable price dispersion and persistent performance differences.

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1 Introduction

Within industrial organisation, price formation is typically analysed through the lens of Bertrand competition. Arguably the most appealing feature of a Bertrand-Nash equilibrium in pure strategies, is ex-post stability. Once all prices are revealed to the market, no firm has an incentive to unilaterally alter its price. Ex-post stability provides an intuitive rationale for using a one-shot Bertrand price-setting game to analyse prices that persist over a relatively long time-horizon.

There are, however, many models of competition for which pure-strategy Bertrand-Nash equilibria cannot be found. Prominent examples include the switching-cost model (Shilony, 1977), Hotelling competition with linear transport costs and closely spaced firms (d’Aspremont, Gabszewicz and Thisse, 1979), and monopolistic competition with captive consumers (Varian, 1980). While Bertrand-Nash equilibria in mixed strategies do exist for these models, the resulting prices are not ex-post stable. Upon observing the prices that emerge from a mixed-strategy equilibrium, it must be the case that at least one firm would gain by unilaterally altering its price.

Several authors have identified the need for an alternative equilibrium concept; one that is consistent with the Bertrand-Nash equilibrium where it exists in pure strategies, but that is capable of identifying ex-post stable prices elsewhere. Shy (1996, Ch. 7) (see also Shy, 2002; Morgan and Shy, 2013) argues that prices should be considered ex-post stable if no firm can profitably undercut its rivals. Shy calls the firm preferred undercut-proof prices, the undercut-proof equilibrium. Similar equilibrium concepts have been propose by Davis, Murphy and Topel (2004) and Iskakov and Iskakov (2012).

While the undercut-proof equilibrium is certainly intuitively appealing, hitherto it has lacked a robust theoretical foundation. A natural critique of the undercut-proof equilibrium, and similar equilibrium concepts, is that they are motivated by assumptions on timing and choice sets, that are not then incorporated into the formal model. Consequently, it is not readily apparent under what conditions the equilibrium concept is valid.

The purpose of this paper is to provide a theoretical foundation for the undercut-proof equilibrium. The approach adopted here is to take, as given, the set of outcomes that are feasible in Bertrand competition. No additional restrictions are placed on the choices available to firms and consumers. Neither are payoff functions altered in any way. The only difference with previous approaches is the concept of ex-post stability employed. Specifically, prices are regarded as ex-post stable if no subset of market participants can improve their respective payoffs by leaving the market, and trading amongst themselves.

This definition of ex-post stability describes the core of a non-transferable utility...
(NTU) cooperative-game, played on the set of outcomes that are feasible in Bertrand competition. Hamilton, MacLeod and Thisse (1991) used the core in this way to determine optimal location choice on the Hotelling line with anonymous consumer locations. Similar techniques have been used elsewhere in the literature to identify stable tax rates (Guesnerie and Oddou, 1981), and natural monopoly prices (Spulber, 1986).

The primary contribution of this paper is to show that, for a broad class of spatial model, the set of prices that give rise to core outcomes is exactly the set of undercut-proof prices. This result supports the use of the undercut-proof equilibrium in switching-costs models with arbitrary numbers of firms and consumers. Moreover, it extends to many familiar models of spatial competition, as well as competition in network markets.

Yet this is not a general result. It depends critically on two conditions: First, firms must have unlimited capacity and constant marginal costs. This prevents consumers from coming into competition with one and other for scarce output; a scenario which would create upward pressure on prices. Second, the goods produced by firms must only be differentiated by the spatial characteristics of the market. If this is the case, complementarities between firms do not arise, and consumers want to trade with at most one firm.

The paper proceeds as follows: The structure of the general spatial model is presented in section 2. Because this model is more general than its precursors, the definitions of undercut-proof prices and equilibria, are likewise generalised. Typically, the set of undercut-proof equilibria will be a subset of the finite maximal elements of the set of undercut-proof prices. A tractable program for identifying the maximal undercut-proof prices is also developed.

The central theoretical contribution of the paper, the equivalence between the set of undercut-proof prices and the core of the NTU cooperative game, is established in section 3.

A Hotelling line with finite consumers is analysed in section 4. While the set of undercut-proof prices for this model contains multiple maximal elements, the two firms share common preferences over these prices, and the equilibrium profits are unique. This example also demonstrates a potential hazard of using a continuum to approximate a large but finite number of consumers. It is shown that where the unit line is home to at least six consumers, the undercut-proof equilibrium prices are strictly lower than is the case with a unit mass.

Section 5 applies the undercut-proof equilibrium to competition on buyer-seller networks. This setting elegantly demonstrates how the undercut-proof equilibrium can be used to describe stable price dispersion. Necessary and sufficient conditions are derived for a component of a network to price competitively. On the remainder of the network it is shown that symmetric firms may have asymmetric equilibrium prices and profits.

The paper concludes with a discussion of the results and some possible extensions.
2 The General Model

The market structure analysed in this paper is simple, yet general enough that it nests a number of prominent models of imperfect competition.

2.1 Market Structure

Consider a market in which a finite set $F$ of firms compete to sell a homogeneous good to a finite set $I$ of consumers. Each firm has unlimited capacity and a constant marginal cost $c_f \geq 0$, that may vary between firms. The terms of trade in the market are described by the price vector $p = \{p_f\}_{f \in F}$, where $p_f$ is the price charged by firm $f$.

On the other side of the market, each consumer has unit demand and values the good at an amount $v > 0$. In addition, a consumer may experience disutility when trading with a given firm. The disutility consumer $i$ experiences when purchasing from firm $f$ is captured by the transaction cost $k_{i,f} \geq 0$. Each consumer purchases from the firm that offers the best price, inclusive of transaction costs. Consumer $i$’s utility, written as a function of the prevailing price vector $p$, is thus,

$$u_i(p) \equiv \max \left\{ \max_{f \in F} [v - (p_f + k_{i,f})], 0 \right\}.$$  

(1)

The utility function has a lower bound of zero because a consumer always has the option of not trading with any firm. The vector of consumer utilities is written $u(p) = \{u_i(p)\}_{i \in I}$.

While the main results in this paper are developed for arbitrary non-negative values of $k_{i,f}$, it is worthwhile illustrating how a number of important market structures are nested within the general model:

1. In the **switching-costs model**, consumers incur a cost $s$ when switching between firms. It follows that $k_{i,f} = 0$ if $i$ has an established relationship with $f$, and $k_{i,f} = s > 0$ otherwise.

2. In a spatial model $k_{i,f}$ represents the cost to $i$ of travelling to $f$. For example, in the **Hotelling model** discussed in section 4, $k_{i,f}$ is equal to the unit travel-cost $t$ times the distance between $i$ and $f$.

3. In the **network market** discussed in section 5, $k_{i,f} = 0$ if there is a link connecting $i$ to $f$, and $k_{i,f} = \infty$ otherwise.

Let $I_f(p) \subseteq I$ be the subset of consumers who purchase from $f$ when the prevailing price vector is $p$. Consumer $i$ only purchases from firm $f$ if $p_f + k_{i,f} \leq p_g + k_{i,g}$ for all $g \in F$. In the event of a tie it is assumed that each consumer makes an arbitrary choice between the best offers.$^2$

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$^2$The results developed in this paper are not sensitive to the tie-breaking rule.
It is now possible to write firm profit as a function of the terms of trade in the market. If the prevailing price vector is $p$, firm $f$’s profit is,

$$\pi_f(p) \equiv (p_f - c_f)\|I_f(p)\|,$$

where the term $\|I_f(p)\|$ refers to the cardinality of the set $I_f(p)$. The vector of firm profits is written $\pi(p) = \{\pi_f(p)\}_{f \in F}$.

### 2.2 Generalising the Undercut-Proof Equilibrium

Shy (1996, Ch. 7) defines the set of undercut-proof prices for a two-firm switching-costs model. The definition readily generalises.

**Definition 1.** The price vector $p^u$ is *undercut-proof* if and only if,

(a) $p_f^u \geq c_f$ for all $f \in F$ and,

(b) $\pi_f(p^u) \geq \pi_f(p_{-f}^u, p_f)$ for all $f \in F$ and $p_f \in [c_f, p_f^u]$.

The set of undercut-proof price vectors is denoted $P^u$.

In words, a price vector is undercut-proof if and only if no firm can strictly increase its profit by unilaterally reducing its price. This is, of course, a necessary (but not sufficient) condition for a pure strategy Bertrand-Nash equilibrium. It follows that if a pure strategy Bertrand-Nash equilibrium exists, it must lie within $P^u$.

It is straightforward to see that $P^u$ is not empty. Each firm charging a price equal to its marginal cost is always undercut-proof. However, the set of undercut-proof price vectors can be very large. Shy (1996, Ch. 7) suggests that the appropriate refinement is to identify the firm preferred price vectors. That is, the price vectors that delivers firms the highest profits.\(^3\)

**Definition 2.** The price vector $p^* \in P^u$ constitutes an *undercut-proof equilibrium* if and only if $\pi(p^*)$ is a maximal element of the set $\{\pi(p^u) : p^u \in P^u\}$.

### 2.3 Properties of Undercut-Proof Equilibria

Shy (1996, Ch. 7) shows that the unique undercut-proof equilibrium of the switching-costs model is the maximum element of the set of undercut-proof prices. The uniqueness of the undercut-proof equilibrium does not extend to the general model, a fact illustrated by examples in sections 4 and 5. However, monotonicity of the profit function on $P^u$ is maintained in the general model.

**Proposition 1.** *Suppose the prevailing price vector is $p^u \in P^u$.*

\(^3\)Hamilton et. al. (1991), Davis et. al. (2004) and Iskakov and Iskakov (2012) employ similar refinements.
(a) If there exists \( \hat{p}^u \in P^u \) such that \( p^u \geq \hat{p}^u \), then \( \pi(p^u) \geq \pi(\hat{p}^u) \).

(b) If \( p^u \) is a maximal element of \( P^u \) then,

\[
p^u_f = \min \left\{ \infty, \min_{i \in I_f(p^u)} [v - k_{i,f}] \right\}
\]

\[
\min_{i \in I_f(p^u)} \left\{ \frac{\|I_g(p^u)\|}{\min_{p_g < p^u} \|I_g(p^u, p_g)\|} + k_{i,g} - k_{i,f} + c_g \right\}
\]

for all \( f \in F \).

(c) There are finite maximal elements of \( P^u \).

(d) If there exists a consumer \( i \in I \), and firms \( f, g \in F \), such that \( i \in I_f(p) \) and \( p^u_f + k_{i,f} = p^u_g + k_{i,g} \), then \( p^u_g = c_g \).

\textbf{Proof.} For (a), it follows from definition 1 that,

\[
\pi_f(p^u) \geq \pi_f(p^u_f, \hat{p}^u_f),
\]

for all \( f \in F \). Moreover, \( p^u_f \geq \hat{p}^u_f \) implies \( I_f(\hat{p}^u) \subseteq I_f(p^u_f, \hat{p}^u_f) \) as firm \( f \) cannot gain customers as a result of rival firms’ discounting. It follows that,

\[
\pi_f(p^u_f, \hat{p}^u_f) = (\hat{p}^u_f - c_f)\|I_f(p^u_f, \hat{p}^u_f)\| \geq (\hat{p}^u_f - c_f)\|I_f(\hat{p}^u)\| = \pi_f(\hat{p}^u).
\]

Thus \( \pi(p^u) \geq \pi(\hat{p}^u) \).

For (b), definition 1 requires that a firm \( g \) does not gain by lowering its price sufficiently to capture a consumer \( i \) from firm \( f \). If \( i \in I_f(p^u) \) then price at which \( i \) becomes indifferent between \( f \) and \( g \) is \( p_g = p^u_f + k_{i,f} - k_{i,g} \). Note that \( g \) may have to price just below this point if \( i \) breaks ties in favour of \( f \). It follows that undercutting to capture \( i \) is not profitable so long as,

\[
(p^u_g - c_g)\|I_g(p^u)\| \geq (p^u_f + k_{i,f} - k_{i,g} - c_g) \min_{p_g \in [c_g, p^u_f + k_{i,f} - k_{i,g}]} \|I_g(p^u, p_g)\|.
\]

Solving for \( p^u_g \) yields the necessary condition,

\[
p^u_g \leq (p^u_g - c_g) \frac{\|I_g(p^u)\|}{\min_{p_g < p^u} \|I_g(p^u, p_g)\|} + k_{i,g} - k_{i,f} + c_g.
\]

Note that, for given values of \( \|I_g(p^u)\| \) and \( \min_{p_g < p^u} \|I_g(p^u, p_g)\| \), (4) is increasing in \( p^u_g \) with a slope in \( (0, 1) \). Moreover, if (4) holds with strict inequality then \( p^u_f + k_{i,f} < p^u_g + k_{i,g} \), implying that \( i \) strictly prefers \( f \)’s price over \( g \). If \( i \in I_f(p^u) \) it must also be the case that,

\[
p^u_f \leq v - k_{i,f}.
\]
On the other hand, if \( I_f(p^u) = \emptyset \) then there is no upper bound on \( f \)'s price thus \( p^u_f \leq \infty \).

By way of contradiction, suppose that \( p^u \) is a maximal element of \( P^u \), and that there exists a firm \( f \) such that \( p^u_f \) is strictly less than the value required by (3). It is sufficient to show that there exists \( \varepsilon > 0 \) such that \((p^u_f, p^u_f + \varepsilon) \in P^u \). There are two ways in which increasing \( p_f \) could cause the price vector to cease to be undercut-proof: First, the increase could change the customer sets in (4). Second, for given customer sets, the price rise could violate the necessary condition in (4).

The composition of the customer sets can be maintained by selecting sufficiently small \( \varepsilon > 0 \). The slack in (4) and (5) then ensures that \( I_g(p^u) = I_g(p^u_f, p^u_f + \varepsilon) \) for all \( g \in F \). Similarly, right-continuity of the integer-valued term \( \min_{p_g < p^u_f + k_i, f - k_i, g} \| I_g(p^u_g, p_g) \| \) in \( p^u_f \) ensures,

\[
\min_{p_g < p^u_f + k_i, f - k_i, g} \| I_g(p^u_g, p_g) \| = \min_{p_g < p^u_f + \varepsilon + k_i, f - k_i, g} \| I_g(p^u_g, p_g) \|,
\]

while for all \( h \notin \{f, g\} \) and \( i \in I_h(p^u) \),

\[
\min_{p_g < p^u_f + k_i, h - k_i, g} \| I_g(p^u_{(f, g)}, p^u_f, p_g) \| = \min_{p_g < p^u_f + k_i, h - k_i, g} \| I_g(p^u_{(f, g)}, p^u_f + \varepsilon, p_g) \|.
\]

If the customer sets are fixed, then for sufficiently small \( \varepsilon > 0 \) the price \( p^u_f + \varepsilon \) continues to satisfies condition (4) with strict inequality for all \( i \in I_f(p^u) \) and \( g \in F \setminus \{f\} \). Moreover, the analogous conditions for all firms \( g \in F \setminus \{f\} \) are weakly relaxed. It follows that \((p^u_{-f}, p^u_{f} + \varepsilon) \in P^u \), a contradiction.

For (c), first note that the cardinalities of the customer sets are integer valued, and lie in the range \( \{0, \ldots, \| I \| \} \). Fix values of \( \| I_g(p^u) \| \) and \( \min_{p_g < p^u_f + k_i, f - k_i, g} \| I_g(p^u_g, p_g) \| \) from within this range. For all \( f \in F \), the constraint in (3) is now the piecewise minimum of linear functions of \( p^u_f \), with slopes in the range \([0, 1] \). It follows that the system described in (b) is a contraction mapping on \( \mathbb{R}^{\| F \|} \). Thus, by the Banach fixed-point theorem there exists a unique solution. Given that \( \| I_g(p^u) \| \) and \( \min_{p_g < p^u_f + k_i, f - k_i, g} \| I_g(p^u_g, p_g) \| \) can take finite values, there are at most finite solutions to (3).

For (d), by way of contradiction suppose that \( p^u_g > c_g \). If firm \( g \) lowers its price by arbitrarily small \( \varepsilon > 0 \) then,

\[
\pi_g(p^u_{-g}, p^u_g - \varepsilon) = (p^u_g - \varepsilon - c_g) \| I_g(p^u_{-g}, p^u_g - \varepsilon) \| \\
\geq (p^u_g - \varepsilon - c_g) (\| I_g(p^u_g) \| + 1) \\
> (p^u_g - c_g) \| I_g(p^u_g) \| \\
= \pi_g(p^u_g)
\]

contradicting the assumption that \( p^u \in P^u \).

Proposition 1(a) states that, within the set of undercut-proof prices, the preferences of firms are (weakly) aligned whenever two price vectors can be ordered. This means that undercut-proof equilibrium profits can always be implemented by a maximal element of
the set of undercut-proof prices. However, as demonstrated in sections 4 and 5, there may exist multiple maximal price vectors and multiple equilibria.

Proposition 1(b) establishes that (3) is a necessary condition for a price vector to be a maximal element of \( P^u \). While proposition 1(c) states that there are finite solutions to (3). It is important to note that the system in (3) is not a sufficient condition. However, it does describe a computationally tractable program that identifies finite candidate equilibria. The example in section 4 demonstrates an application of this program.

Finally, proposition 1(d) states that the only circumstance under which a consumer can be indifferent between two firms, when prices are undercut-proof, is when the firm that misses out on the consumer’s business is pricing at marginal cost. While straightforward, this result is important as it demonstrates that the openness problem from Bertrand competition may also be present in the set of undercut-proof prices.

3 A Theoretical Foundation

The primary contribution of this paper is to show that the set of undercut-proof prices is equivalent to the core of an NTU cooperative game, played on the set of outcomes that are feasible in Bertrand competition.

3.1 Feasible Sets

The first step in analysing the game is to establish the outcomes that are feasible to each subset of market participants. In order to do this, it is first necessary to introduce some additional notation that will be used in this section only.

A coalition is a non-empty subset of market participants \( G \subseteq I \cup F \). The outcomes that are feasible to a coalition \( G \) are the vectors of consumer utilities and firm profits that would be possible if the market were comprised only of the member of \( G \).

Suppose that the prevailing price vector in the coalition is \( p_G = \{p_f\}_{f \in G \cap F} \). From (1) it follows that the utility of a consumer \( i \in G \cap I \) is,

\[
  u_i(p_G|G) = \max \left\{ \max_{f \in G \cap F} [v - (p_f + k_{i,f})], 0 \right\}.
\]

This utility function differs from (1) only in that it restricts \( i \) to purchase from a firm that is likewise a member of \( G \). Similarly, from (2) it follows that the profit of a firm \( f \in G \cap F \) is,

\[
  \pi_f(p_G|G) \equiv (p_f - c_f) || I_f(p_G|G) ||,
\]

where \( I_f(p_G|G) \subseteq G \cap I \) is the subset of consumers in the coalition \( G \) who purchase from \( f \) given the prevailing price vector.

The payoffs to the members of the coalition \( G \) can be summarised by the payoff vector \( \{\pi_G(p_G|G), u_G(p_G|G)\} \), where \( \pi_G(p_G|G) = \{\pi_f(p_G|G)\}_{f \in G \cap F} \) and \( u_G(p_G|G) = \)
{u_i(p_G|G)}_{i \in G \cap I}$. The set of all payoff vectors that are feasible to the coalition $G$ is thus,

$$V(G) \equiv \left\{ \{\pi_G(p_G|G), u_G(p_G|G)\} : p_G \in \mathbb{R}^{\|F \cap G\|} \right\}. \quad (6)$$

In words, the payoff vector $\{\pi_G, u_G\}$ is feasible to a coalition $G$, if and only if there exists a price vector $p_G$, that would support the payoffs in Bertrand competition.

The set $V(G)$ describes both the aggregate gain in social welfare that can be created by transactions between the members of $G$, and the ways in which that welfare can be distributed within the coalition. Of course, a surplus is only generated when firms trade with consumers. For a coalition $G$ that consists entirely of consumers (or firms), the feasible set is equal to the zero vector $V(G) = 0$.

### 3.2 Core Equivalence

The pair $(\{I \cup F\}, V(\cdot))$ defines a non-transferable utility (NTU) coalitional game. The standard solution concept for coalitional games is the core; the set of outcomes (payoff vectors) that are stable in the sense that they cannot be blocked by a coalition of players.

The coalition $G$ can block a payoff vector $\{\pi, u\}$ if the members of $G$ all strictly prefer some alternative price structure $p_G$, and are willing trade at these terms, even if it means excluding the remaining market participants.

**Definition 3.** A payoff vector $\{\pi^c, u^c\} \in V(I \cup F)$ lies in the core of $\{(I \cup F), V(\cdot)\}$ if and only if for all $G \subseteq I \cup F$ and $\{\pi_G, u_G\}$, if $\pi_f > \pi^c_f$ for all $f \in F \cap G$ and $u_i > u^c_i$ for all $i \in I \cap G$, then $\{\pi_G, u_G\} \notin V(G)$. The set of core payoffs is written $\text{core}(\{I \cup F\}, V(\cdot))$.

Note that, by definition, the core is a subset of the payoffs that are feasible when all members of $I \cup F$ participate in the market. Moreover, a payoff vector is only feasible if there exists a price vector that supports it. It follows that is possible to back out the set of price vectors that give rise to core outcomes.

**Definition 4.** If $\{\pi(p^c), u(p^c)\} \in \text{core}(I \cup F, V(\cdot))$ then $p^c$ is a core price vector. The set of core price vectors is written $P^c$.

It is now possible to state the central result of this paper.

**Proposition 2.** $P^c = P^u$.

**Proof.** The proof proceeds via three steps:

**Step 1:** If $p^c \in P^c$ then $p^c \in P^u$. Suppose to the contrary that $p^c \notin P^u$. There exists $f \in F$ and $p_f \in [c_f, p^c_f)$ such that $\pi_f(p^c_f) < \pi_f(p^c_{-f}, p_f)$. Consider the coalition $G = \{f\} \cup I_f(p^c_{-f}, p_f)$. For all $i \in I_f(p^c_{-f}, p_f)$ it must be the case that,

$$u_i(\{p_f\}|G) = u_i(p^c_{-f}, p_f) \geq u_i(p^c),$$
and as such all consumers in \( G \) weakly prefer the price \( p_f \) from firm \( f \) to their most preferred prices in the price vector \( p^c \).

Suppose that the coalition \( G \) deviates at the price \( p_G = \{p_f - \varepsilon\} \). For sufficiently small \( \varepsilon > 0 \) each consumer in \( G \) must be strictly better off as a result of the deviation while \( \pi_f(\{p_f - \varepsilon\})|G| > \pi_f(p^c) \); a contradiction.

**Step 2:** Suppose that \( p^u \in P^u \). The payoff vector \( \{\pi(p^u), u(p^u)\} \) cannot be blocked by a coalition \( G \) where \( \|F \cap G\| = 1 \). Let \( F \cap G = f \). For a consumer \( i \in G \), \( p_f \geq p^f_i \) implies \( u_i(\{p_f\})|G| \leq u_i(p^u) \), thus we need only consider cases in which \( p_f < p^f_i \).

By way of contradiction suppose that there exists \( p_f \in [c_f, p^f_i] \) such that \( \pi_f(\{p_f\})|G| > \pi_f(p^u) \) and \( u_i(\{p_f\})|G| > u_i(p^u) \) for all \( i \in I \cap G \). For a consumer \( i \) to be strictly better off in the deviation it must be the case that \( i \) strictly prefers \( p_f \) to his/her most preferred price in \( p^u \). It follows that \( i \in I_f(p^u - f, p_f) \). But \( \pi_f(p^u - f, p_f) \geq \pi_f(\{p_f\})|G| > \pi_f(p^u) \) contradicting the assumption that \( p^u \) is undercut-proof.

**Step 3:** If a payoff vector \( \{\pi(p), u(p)\} \) can be blocked by a coalition \( \hat{G} \) deviating at the price vector \( \hat{p}_G \). There exists a sub-coalition \( G \subseteq \hat{G} \), with \( \|F \cap G\| = 1 \), that can also block the payoff vector. Select some \( f \in F \cap \hat{G} \) and let \( G = f \cup I_f(\hat{p}_G|\hat{G}) \). By construction \( \pi_f(\hat{p}_f|G) = \pi_f(\hat{p}_G|\hat{G}) \) and for all \( i \in I_f(\hat{p}_G|\hat{G}) \), \( u_i(\hat{p}_f|G) = u_i(\hat{p}_G|\hat{G}) \), thus \( G \) can also block \( \{\pi(p), u(p)\} \) by deviating at the price vector \( p_G = \{\hat{p}_f\} \).

Taken together steps 2 and 3 prove that if \( p^u \in P^u \) then \( p^u \in P^c \).

The intuition behind proposition 2 is straightforward. Where firms have unlimited capacity consumers are never in conflict and therefore have a common interest in seeing prices fall. At the same time, the products of firms do not display complementarities and, therefore, each consumer need only trade with the firm offering the best terms of trade. It follows that the only deviating coalitions that need be considered are coalitions consisting of one firm and the subset of consumers willing to purchase from the firm on the condition that the firm’s price falls; a deviation that is equivalent to a unilateral price reduction in Bertrand competition.

### 4 Hotelling Competition with Finite Buyers

Two of the special cases to which the undercut-proof equilibrium have previously been applied are the switching-costs model (Shy, 1996, Ch. 7) and competition on the Hotelling line with linear transport costs (Hamilton et. al., 1991; Iskakov and Iskakov, 2012). These two models can be regarded at the limiting cases of competition on the Hotelling line with finite evenly spaced consumers. This model also provides an elegant illustration of the multiplicity of maximal undercut-proof prices.
Suppose that the set of consumers is indexed \( I = \{1, 2, \ldots, n\} \), and that consumers are evenly spaced along the unit line such that consumer 1 is located at the left end while consumer \( n \) occupies the right. Each consumer faces a linear transport cost of \( t \) per unit of distance they must travel to reach a firm. Moreover, each consumer has unit demand, and values the good at \( v > 2t \).

Two firms operate in the market with firm 1 collocated with consumer 1 at the left end of the line, and firm 2 collocated with consumer \( n \) at the right. The disutility \( i \) experiences when purchasing from \( f \) is thus,

\[
k_{i,f} = t \left| \frac{i - 1}{n - 1} - (f - 1) \right|.
\]

Both firms have a marginal cost of zero.

**Proposition 3.** The price vector \( p^u \) is undercut-proof if and only if there exists \( i^* \in I \) such that,

\[
0 \leq p_1^u \leq p_2^u \frac{n - i^*}{n - i^* + 1} + t - 2t \frac{i^* - 1}{n - 1}, \tag{7}
\]

and,

\[
0 \leq p_2^u \leq p_1^u \frac{i^*}{i^* + 1} - t + 2t \frac{i^*}{n - 1}. \tag{8}
\]

Where the market contains an even number of consumers the undercut-proof equilibrium is \( p_1^* = p_2^* = t(n + 2)/2(n - 1) \). Where the market contains an odd number of consumers the undercut-proof equilibria are \( p_f^* = t(n + 3)/2(n - 1) \) and \( p_g^* = t(n + 3)/2(n + 1) \) for \( f \neq g \). In either case, undercut-proof equilibrium profits are unique.

**Proof.** First note that a consumer \( i \) is indifferent between the two firms if,

\[
p_1 + k_{i,1} = p_1 + t \frac{i - 1}{n - 1} = p_2 + t \left(1 - \frac{i - 1}{n - 1}\right) = p_2 + k_{i,2},
\]

implying \( p_1 = p_2 + t - 2t \frac{i - 1}{n - 1} \). The proof now proceeds in four steps.

**Step 1:** Conditions (7) and (8) are necessary for undercut-proof prices. Suppose that \( p^u \in P^u \), \( i^* \in I^1(p^u) \) and \( i^* + 1 \in I^1(p^u) \). For firm 1 to capture consumer \( i^* + 1 \) it must drop its price to (just below) \( p_1 = p_2^u + t - 2t \frac{i^*}{n - 1} \). Firm 1 will not gain from this deviation if,

\[
p_1^u i^* \geq (i^* + 1) \left(p_2^u + t - 2t \frac{i^*}{n - 1}\right),
\]

which in turn implies (8). Similarly for firm 2 to capture consumer \( i^* \) it must drop its price to (just below) \( p_2 = p_1^u - t + 2t \frac{i^* - 1}{n - 1} \). Firm 2 will not gain from this deviation if,

\[
p_2^u (n - i^*) \geq (n - i^* + 1) \left(p_1^u - t + 2t \frac{i^* - 1}{n - 1}\right),
\]

which in turn implies (7).
Step 2: For all $i^* \in \left[\frac{n-1}{4}, \frac{3n+1}{4}\right] \cap \{1, \ldots, n-1\}$ the maximal price vectors satisfying (7) and (8) are,

$$p^u_1 = t \frac{(3n - 4i^* + 1)(i^* + 1)}{n^2 - 1} \quad \text{and} \quad p^u_2 = t \frac{(4i^* - n + 1)(n+1 - i^*)}{n^2 - 1}. \quad (9)$$

The maximal price pair for a given $i^*$ must simultaneously satisfy the right inequalities of (7) and (8) with equality. Substituting for $p^u_2$ into (7) yields,

$$p^u_1 = \frac{(p^u_1 i^* + 1 - t + 2t \frac{i^*}{n-1})}{n-1} \frac{n-i^*}{n-i^*+1} + t - 2t \frac{i^*-1}{n-1},$$

collecting like terms and solving for $p^u_1$ yields the required result. Substituting for $p^u_1$ into (8),

$$p^u_2 = \frac{t (3n - 4i^* + 1)(i^* + 1)}{n^2 - 1} \frac{i^*}{i^*+1} - t + 2t \frac{i^*}{n-1},$$

simplifying yields the required result. The price vector $p^u \in \mathbb{R}_+^2$ when $i^* \in \left[\frac{n-1}{4}, \frac{3n+1}{4}\right] \cap \{1, \ldots, n-1\}.$

Step 3: Neither firm is willing to drop its price to capture more than one consumer where conditions (7) and (8) hold. Consider a price vector $p^u$ satisfying (7) and (8). Firm 1 will not lower its price to capture $m \in \{2, \ldots, n-i\}$ additional consumers if,

$$p^u_1 i^* \geq (i^* + m) \left( p^u_2 + t - 2t \frac{i^* + m - 1}{n-1} \right),$$

which in turn implies,

$$p^u_1 i^* \geq (i^* + 1) \left( p^u_2 + t - 2t \frac{i^*}{n-1} \right) + (m - 1) \left( p^u_2 + t - 2t \frac{2i^* + m}{n-1} \right). \quad (10)$$

From (8) it follows that (10) must hold if,

$$p^u_2 \leq t \left( \frac{4i^* + 2m}{n-1} - 1 \right),$$

substituting for the maximum value of $p^u_2$ from (9),

$$t \frac{(4i^* - n + 1)(n+1 - i^*)}{n^2 - 1} \leq t \left( \frac{4i^* + 2m}{n-1} - 1 \right),$$

and simplifying,

$$4i^* + 2 \frac{m}{i^*} (n+1) \geq n - 1.$$

This condition must hold for $p^u$ satisfying (7) and (8) as the lower bound on $i^*$ is $(n-1)/4.$

Step 4: Establish the firm preferred price vectors. The market share $i^* : n - i^*$ that maximises firm 1’s profit at the maximal price vector solves,

$$i^*_1 = \arg\max_{i \in \{1, \ldots, n-1\}} t \frac{i(3n - 4i + 1)(i+1)}{n^2 - 1},$$

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Figure 1: Undercut-Proof Prices

Table 1: Undercut-Proof Equilibria

<table>
<thead>
<tr>
<th>n</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>20</th>
<th>n → ∞</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1^* = p_2^* )</td>
<td>2t</td>
<td>t</td>
<td>( \frac{4}{5}t )</td>
<td>( \frac{5}{7}t )</td>
<td>( \frac{2}{3}t )</td>
<td>( \frac{11}{15}t )</td>
<td>( \frac{1}{2}t )</td>
</tr>
</tbody>
</table>
while for firm 2 the argument solves,

\[ i^*_2 = \argmax_{i \in \{1, \ldots, n-1\}} \frac{t}{n^2 - 1} (n - i)(4i - n + 1)(n + 1 - i). \]

Where \( n \) is even \( i^*_1 = i^*_2 = n/2 \) and where \( n \) is odd \( i^*_1 = i^*_2 = \{(n - 1)/2, (n + 1)/2\} \). Substituting these values into (9) yields the required result. □

Proposition 3 is illustrated in figure 1. Panel (a) shows the set of undercut-proof prices when there are two consumers in the market, collocated with the two firms. This setup is equivalent to the switching-costs model. The maximum element of \( P^u \), and the unique undercut-proof equilibrium, is \( p^*_1 = p^*_2 = 2t \). Figure 1(a) also illustrates proposition 1(d). The dotted lines represent the set of prices for which one or other consumer is indifferent between the two firms. Notice that the set \( P^u \) only intersects with these lines where one or other of the firms is pricing at marginal cost.

Panel (b) of figure 1 shows how the set of ex-post stable outcomes changes when a third consumer is present at the centre of the line. The set of undercut-proof prices is split in two by the set of prices at which the centrally located consumer is indifferent between the two firms. There are now two maximal undercut-proof price vectors, \((\frac{3t}{4}, \frac{3t}{4})\) and \((\frac{3t}{4}, \frac{3t}{4})\), both of which are undercut-proof equilibria. However, firm profits are identical in both equilibria.

The three consumer case is an example of ex-post stable price dispersion. The two firms are identical, and occupy symmetric positions in the market. In equilibrium, one firm must adopt a low margin, high market share strategy, while its opponent does the opposite.

The four consumer case is illustrated in panel (c). The set of undercut-proof prices vectors now has three maximal elements. However, only one of these \((t, t)\), corresponds to an undercut-proof equilibrium as it delivers both firms earn higher profits than either of the alternatives. Panel (d) shows the six consumer case. Again there are three maximal elements, only one of which is an equilibrium. Of interest here is the fact that some market divisions are never ex-post stable. There is no undercut-proof price vector that will deliver five of six consumers to a particular firm.

Table 1 lists the undercut-proof equilibria for various numbers of consumers. Notice that the equilibrium prices are decreasing in the number of consumers in the market. Moreover, in the limit, as the number of consumers becomes arbitrarily large, the equilibrium prices converge \( p^*_1 = p^*_2 = t/2 \). By contrast, Hamilton et. al. (1991) (see also Iskakov and Iskakov, 2012) prove that where the Hotelling model is populated by a uniformly distributed continuum of consumers, the undercut-proof equilibrium is exactly the Bertrand-Nash equilibrium \( p^*_1 = p^*_2 = t \). In this setting at least, a continuum of consumers is not a good approximation for markets with large, but finite, numbers of consumers.

The intuition behind the discrepancy is straightforward. Consider a market with a continuum of consumers: In order to capture an additional fraction 1/n of consumers a
firm must lower its price to move the point of indifference a distance \(1/n\) closer to its rival. Now consider the case in which the unit line is populated by \(n\) discrete consumers. In an undercut-proof equilibrium, the point of indifference is equidistant between two consumers. In order to capture the marginal consumer a firm must drop its price, moving the point of indifference up to (or just past) this consumer; a distance of \(1/2(n - 1)\). It follows that as \(n \to \infty\) capturing the marginal consumer is half as costly, and consequently the highest undercut-proof price vector is half as large. This result suggests that continua of consumers should be used with caution in applied work.

5 Price Competition in Network Markets

Network markets are an elegant way of modelling competition in which consumers can only purchase from a subset of firms. A buyer-seller can capture features such as customer lock-in, product compatibility, and even purchasing habits. In common with the other special cases examined in this paper, network markets do no typically possess Bertrand-Nash equilibria in pure strategies (see for example Guzman, 2011).

Consider a market in which the disutility a consumer experiences when trading with a firm is either zero or infinity, and all firms have a marginal cost of zero. Consumer \(i\) is said to be linked to firm \(f\) if \(k_{i,f} = 0\). Conversely, no transaction between \(i\) and \(f\) is possible if \(k_{i,f} = \infty\). The pattern of links describes a buyer-seller network.

**Definition 5.** For a consumer \(i\), the linked set \(L(i) = \{f \in F : k_{i,f} = 0\}\). The linked set for a firm \(f\) is defined analogously.

The linked set \(L(i) \subseteq F\) is the set of firms from which consumer \(i\) can purchase. It is the set of firms in competition for \(i\)’s business. Similarly, the linked set \(L(f) \subseteq I\) is the set of \(f\)’s potential customers. Linked sets can also be defined for a set of firms \(\hat{F} \subseteq F\) such that \(L(\hat{F}) = \bigcup_{f \in \hat{F}} L(f)\). It is convenient to assume that \(L(i) \neq \emptyset\) for all \(i \in I\), and \(L(f) \neq \emptyset\) for all \(f \in F\).

5.1 The Two Components of a Network Market

The network market structures gives rise to ex-post stable price dispersion in two ways: First, as described in the following proposition, firms on one component of a network can enjoy market power, while on the remainder of the network conditions are perfectly competitive. Second, as demonstrated in a subsequent example, firms in symmetric locations on the non-competitive component of the network may have different equilibrium prices and profits.

\(^4\)Kranton and Minehart (2000, 2001) and Corominas-Bosch (2004) have characterised pure strategy solutions for markets in which each firm has a single indivisible unit to sell. They utilise bargaining protocols to solve for prices. Unfortunately, their results do not generalise to price competition amongst firms with unlimited capacities.
Definition 6. A set of firms $\hat{F} \subseteq F$ is competitive if and only if, every consumer linked to one firm in $\hat{F}$ is also linked to a second firm in $\hat{F}$ (formally, if $L(i) \cap F^c \neq \emptyset$ then $\|L(i) \cap F^c\| \geq 2$). The union of all competitive sets of firms is written $F^c$.

Proposition 4. Suppose that $p^u \in P^u$.

(a) If $L(i) \cap F^c \neq \emptyset$ then $\min_{f \in L(i)} p^u_f = 0$.

(b) If $p^u$ is an undercut-proof equilibrium then $p^u_f > 0$ and $\pi_f(p^u) > 0$ for all $f \in F \setminus F^c$.

Proof. For (a), by way of contradiction suppose that there exists a consumer $j$ such that $L(j) \cap F^c \neq \emptyset$ and $\min_{f \in L(i)} p^u_f > 0$. Take the consumer,

$$i = \operatorname{argmax}_{j \in L(F^c)} \left[ \min_{g \in L(i)} p^u_g \right].$$

By definition 6, $\|L(i) \cap F^c\| \geq 2$. It follows that there exists $f \in L(i) \cap F^c$ such that $i \notin I_f(p^u)$. But for sufficiently small $\varepsilon > 0$,

$$\pi_f(p^u) \leq \min_{g \in L(i)} p^u_g \|I_f(p^u)\| < \left( \min_{g \in L(i)} p^u_g - \varepsilon \right) \|I_f(p^u) \cup \{i\}\| \leq \pi_f \left( p^u_{-f}, \min_{g \in L(i)} p^u_g - \varepsilon \right),$$

a contradiction.

For (b) by way of contradiction, suppose that there exists $g \in F \setminus F^c$ such that $\pi_g(p^u) = 0$. Let $F^0 = \{g \in (F \setminus F^c) : \pi_g(p^u) = 0\}$. By definition 6 there exists a consumer $i$ and firm $f \in F^0$, such that $L(i) \cap (F^0 \cup F^c) = \{f\}$. It follows from the selection of $i$ and the construction of $F^0$ that $p^u_f = 0$ and $p^u_g > 0$ for all $g \in L(i) \setminus \{f\}$.

Consider the price vector $p$ such that $p_g = p^u_g$ for all $g \in F \setminus (F^0 \cup F^c)$, $p_g = 0$ for all $g \in (F^0 \cup F^c) \setminus \{f\}$, and $p_f = \varepsilon > 0$. For sufficiently small $\varepsilon > 0$, $I_f(p) = L(f) \setminus L((F^0 \cup F^c) \setminus \{f\}) \neq \emptyset$, and $I_g(p) = I_g(p^u)$ for all $g \in F \setminus (F^0 \cup F^c)$. It follows that $\pi_f(p) > 0$, and $\pi_g(p) = \pi_g(p^u)$ for all $g \in F \setminus (F^0 \cup F^c)$. Given that $p^u \in P^u$, the price vector $p$ is undercut-proof no firm in $F \setminus (F^0 \cup F^c)$ is willing to lower its price to capture consumers in $I_f(p)$. This will be the case if $\varepsilon$ is sufficiently close to zero. But $\pi(p) \geq \pi(p^u)$ with strict inequality for $\pi_f(\cdot)$, contradicting the assumption that $p^u$ is an undercut-proof equilibrium.

The intuition behind proposition 4(a) is straightforward. Consider the component of the network consisting of the competitive set $F^c$, and those consumers connected to the competitive set $L(F^c)$. On this component there are always at least two firms competing for each consumer. Moreover, no firm in $F^c$ has as an outside option, the opportunity to sell to a consumer outside of this component. Thus competition between the firms in $F^c$ is sufficiently fierce to ensure that every consumer linked to these firms can access the good at marginal cost.
An immediate corollary of proposition 4(a) is that $F^c = F$ if all buyers are linked to at least two sellers. For a component on the network to be non-competitive, there must exist a consumer who can only trade with a single firm.

The second part of proposition 4 establishes that, in an undercut-proof equilibrium, all firms on the remainder of the network have sufficient market power that they price above marginal cost and enjoy strictly positive profits. The intuition behind the proof is derived from the structure of the non-competitive component of the network. Take any subset of firms on this component and at least one firm must be linked to a consumer who is not then linked to a second firm, or who is only linked to firms with positive prices. It follows that the firm can select a price that is greater than marginal cost, but low enough to be undercut-proof, capture the consumer and earn positive profits.

Proposition 4 is remarkably robust. Proposition 4(a) also holds for a Bertrand-Nash equilibrium in the network market. While logic underlying the proof of proposition 4(b) can be applied to prove that all firms in $F^c$ will have a strictly positive expected payoff on the mixed strategy Bertrand-Nash equilibrium.

5.2 Persistent Performance Differences of Actually Identical Enterprises

While the Hotelling line example in section 4 illustrates how the undercut-proof equilibrium can explain ex-post stable price dispersion in a market, it is important to note that firms had identical equilibrium profits. This is not generally the case. A remarkable feature of the non-competitive component of a network market is that identical firms, with symmetric locations on the network, may necessarily have different equilibrium prices and profits.

Consider the network market illustrated in figure 2. The market consists of three firms and three consumers. Firm 2 can sell to all three consumers, while firms 1 and 3 are in competition with firm 2 for consumers 1 and 3 respectively. Applying definition 6, it is clear that this network does not contain a competitive component. Within every subset

\footnote{It is possible for firms on the Hotelling line to have asymmetric equilibrium profits. However, this requires the firms to have asymmetric locations or costs.}
of firms, there exists at least one firm, linked to a consumer, who is not then linked to another firm in the subset. It follows from proposition 4 that, in equilibrium, each firm has a positive price and earns a positive profit.

Firms enjoy market power on this network because consumer 2 can only purchase from firm 2. In equilibrium, firm 2 sets the monopoly price of \( v \) to maximise the rents it extracts from its captive consumer. Firm 2’s profit creates the opportunity for firms 1 and 3 to profitably charge prices in excess of marginal cost. The only constraints on \( p_1 \) and \( p_3 \) are they must be sufficiently low that firm 2 does not find it profitable to undercut to capture additional consumers. In fact, there are two undercut-proof equilibria for this market:

\[
(p_1^*, p_2^*, p_3^*) = \left( \frac{v}{2}, v, \frac{v}{3} \right) \quad \text{and} \quad (p_1^*, p_2^*, p_3^*) = \left( \frac{v}{3}, v, \frac{v}{2} \right).
\]

With marginal cost of zero and one sale per firm, equilibrium firm profits are equal to firm prices. The highest of \( p_1^* \) and \( p_3^* \) is \( v/2 \); the price at which it is no longer profitable to undercut to capture a single consumer. The remaining price \( v/3 \) prevents profitably capturing both contested consumers.

Despite firms 1 and 3 occupying symmetric positions on the network, the two firms must have different equilibrium prices and profits. Effectively, one firm is pricing to protect its own customer while the other must price to protect both contested consumers.

6 Discussion

This paper generalises the undercut-proof equilibrium to a rich spatial setting that nests a number of important market structures. A robust theoretical foundation is developed, based on the premise that a market outcome is ex-post stable if no subset of market participants can gain by leaving the market and trading amongst themselves. Applications of the undercut-proof equilibrium to the Hotelling line and network competition illustrate the ease of use of this solution concept in more general settings, and show how stable price dispersion and persistent performance differences can be explained in simple models.

6.1 Ex-Post Stability

The theoretical foundation of this paper is built on the premise that a firm must bring consumers with it in a deviation.

There is a significant body of experimental evidence suggesting that consumers are resistant to price rises that do not reflect cost increases and may express their displeasure limiting their purchases (see for example Kahneman, Knetsch and Thaler, 1986a,b; Kujala and Smith, 2008). Many firms appear to be cognisant of this behaviour, going out of their way to identify price rises resulting from cost shocks. Clear examples of the signalling of a cost shock include the fuel surcharge on airline tickets during periods of high oil prices,
and warnings by retailers of price increases following bad crops or crop losses. Taken together these factors suggest that, at least in some markets, consumers play a more than passive role in determining price levels.

Which leads to the question of coalition formation. Within the NTU cooperative game, a blocking coalition in a subset of market participants with a common incentive to change the terms of trade in the market. In general, a blocking coalition must encompass both sides of the market. Without both firms and consumers the coalition would be unable to generate the surplus necessary to improve the payoffs of all deviating players. It follows coalition formation can be achieved by a firm identifying an appropriately sized subset of consumers who will benefit from altering the terms of trade.

Firms operating in oligopoly markets are typically very effective at locating customers. They can gather significant information through the operation of store accounts and loyalty programs, and may offer preferred customers preferential access to new or limited stock.

6.2 Further Generalisation of the Undercut-Proof Equilibrium

While the model developed in this paper is substantially more general than the models to which the undercut-proof equilibrium has previously been applied, propositions 1 and 2 generalise further still. For example, downward sloping demand can accommodated by utilising a richer utility function,

\[ u_i(p) = \max \left\{ \max_{f \in F} \nu_i^{-1}(p_f + \kappa_{i,f}) - (p_f + \kappa_{i,f}) \nu_i^{-1}(p_f + \kappa_{i,f}) - k_{i,f}, 0 \right\}, \]

where \( \nu(\cdot) \) is continuous, weakly concave increasing, and normalised to zero if nothing is consumed.\(^6\) The disutility \( i \) experiences in purchasing from \( f \) comes in two parts; a fixed component \( k_{i,f} \), and an additional disutility per unit purchased \( \kappa_{i,f} \).

Another straightforward generalisation is to allow multiple firm locations, each with its own price. Suppose that firm \( f \in F \) occupies a number \( n \) of locations. Firm \( f \)'s price vector can be expressed as \( p_f = (p_{f,1}, \ldots, p_{f,n}) \). In this case if the undercutting requirement \( p_j \in [c_j, p_j^u] \) in definition 1 is read as \( p_{f,m} \in [c_j, p_{f,m}^u] \) for all \( m = 1, \ldots, n \) then proposition 2 continues to hold.

References


\(^6\)It follows that \( \nu_i^{-1}(\cdot) \) is \( i \)'s demand function.


P. Morgan & O. Shy (2013), Undercut-Proof Equilibria with Multiple Firms, Mimeo.


