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Likelihood Ratio Tests for a Unit Root in Panels with Random Effects

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Abstract

Because of the fixed heterogeneity of their models, most panel unit root tests impose restrictions on the rate at which the number of time periods, $T$, and the number of cross-section units, $N$, go to infinity. A common example of such a restriction is $N/T \to 0$, which in practice means that $T >> N$, a condition that is not always met. In the current paper the heterogeneity is given a parsimonious random effects specification, which is used as a basis for developing a new likelihood ratio test for a unit root. The asymptotic analysis shows that the new test is valid for all $(N, T)$ constellations satisfying $N/T^5 \to 0$, which represents a substantial improvement when compared to the existing fixed effects literature.

JEL Classification: C12; C13; C33.

Keywords: Unit root; panel data; random effects.

1 Introduction

Consider the following dynamic panel data model:

$$y_{it} = \rho y_{i,t-1} + \gamma + \mu_i + \lambda_t + \nu_{i,t},$$ (1)
where \( t = 1, ..., T \) and \( i = 1, ..., N \) index the time series and cross-section units, respectively, 
\[ y_{1,0} = ... = y_{N,0} = 0, \gamma \] is a fixed constant, \( \mu_i \) and \( \lambda_t \) are unit- and time-specific effects, respectively, and \( \nu_{i,t} \) is an error term. There is by now a burgeoning literature dealing with the problem of how to test for a unit root (\( \rho = 1 \)) in such models (see Breitung and Pesaran, 2008; Baltagi, 2008, Chapter 12, for surveys). The standard practice is to treat \( \mu_i \) and \( \lambda_t \) as fixed effects (FE) to be estimated along with the other parameters of the model. FE models are not parsimonious, however. In fact, since in this literature it is customary to assume that \( N, T \to \infty \), \( \mu_1, ..., \mu_N \) and \( \lambda_1, ..., \lambda_T \) are incidental parameters, and the challenges posed by such parameters are particularly acute in dynamic panels. The most well-known challenge by far is the presence of bias, for which there is a large literature (see Moon et al., 2013, for a survey). Specifically, the problem (under \( N, T \to \infty \) asymptotics) is that while the least squares (LS) estimator of \( \alpha \) is consistent, its asymptotic distribution is not correctly centered, which complicates inference. The conventional approach by which researchers have been dealing with this problem is to bias correct. For such corrections to be successful, however, it is necessary to control the relative expansion rate of \( N \) and \( T \), for otherwise the (correction-induced) approximation error will be non-negligible.

A common requirement is that \( N/T \to 0 \) (see, for example, Bai and Ng, 2010; Levin et al., 2002; Moon and Perron, 2004; Moon et al., 2007; Westerlund, 2014, 2015), which in practice means that \( T > N \). Many unit root tests are therefore limited in their applicability. An extreme example is provided by Demetrescu and Hanck (2012), who assume that \( N^5/T \to 0 \), which in practice means \( T > N^5 \). Hence, for tests developed under this condition to be applicable if, for example, \( N = 10, T > 100,000 \) is required. It is therefore not surprising to find that the tests of Demetrescu and Hanck (2012) tend to behave very poorly in samples of realistic size (see Westerlund, 2014). As Westerlund and Breitung (2013) demonstrate, however, one does not have to go to such extremes for \( N/T \) to have an effect. Indeed, as a large body of Monte Carlo evidence shows (see, for example, Wagner and Hlouskova, 2006; Gengenbach et al., 2009, for large-scale simulation studies), \( N/T \) is absolutely crucial in determining the (relative) performance of tests, so much so that researchers might well find themselves discarding data in order to have \( N \) sufficiently small relative to \( T \).

The current paper is motivated by these observations. The purpose is to develop a maximum log-likelihood ratio (LR) test for testing \( H_0: \rho = 1 \) and \( \gamma = 0 \) versus \( H_1: |\rho| < 1 \) when \( \mu_i \) and \( \lambda_t \) are random. This random effects (RE) assumption holds considerable promise in
that it greatly reduces the number of parameters that need to be estimated. This is in turn expected to lead to a reduction in bias, thereby lessening the dependence on $N/T$.

The analysis is conducted within the context of a simple but transparent model with serially uncorrelated errors, which is the kernel of most dynamic econometric models and its properties are fundamental to more complicated models. In spite of this simplicity, however, the maximum likelihood approach adopted here turns out to be very challenging, yet rewarding. Indeed, we show that under the RE assumption the LR test statistic lends itself to simple chi-squared inference for all $N/T$ ratios satisfying $N/T^5 \to 0$ as $N, T \to \infty$, which represents a substantial improvement upon the otherwise so common $N/T \to 0$ condition. Results from a small Monte Carlo study are reported to suggest that the asymptotic results are borne out very well in small samples, with the LR statistic having only minimal distortions across a range of empirically relevant sample sizes, from “micro panels” with $N = 1,000$ and $T = 10$ to “macro panels” with $N = 5$ and $T = 100$.

2 The LR test

Let us define $\varepsilon_{i,t} = \mu_i + \lambda_t + \nu_{i,t}$ and $\alpha = \rho - 1$, such that (1) can be rewritten as
\[
\Delta y_{i,t} = \alpha y_{i,t-1} + \gamma + \varepsilon_{i,t},
\]
or, in matrix notation,
\[
\Delta y = \alpha y_\gamma + \iota_{NT}\gamma + \varepsilon,
\]
where $\iota_{NT} = (1, \ldots, 1)'$ is a $NT \times 1$ vector of ones, $y_\gamma = (y_{1,0}, \ldots, y_{1,T-1}, \ldots, y_{N,0}, \ldots, y_{N,T-1})'$ and $\varepsilon = (\varepsilon_{1,1}, \ldots, \varepsilon_{1,T}, \ldots, \varepsilon_{N,1}, \ldots, \varepsilon_{N,T})'$. In this notation, the null and alternative hypotheses of interest are $H_0 : \alpha = \gamma = 0$ and $H_1 : \alpha < 0$, respectively. Initially, we will assume that $\eta_{i,t} = (\mu_i, \lambda_t, \nu_{i,t})'$ satisfies Assumption 1.

**Assumption 1.** $\eta_{i,t}$ is independently and identically distributed (iid) as $N(0_{3 \times 1}, \Sigma_\eta)$, where $\Sigma_\eta = \text{diag}(\sigma_\mu^2, \sigma_\lambda^2, \sigma_\nu^2)$ with $\sigma_\mu^2 > 0$ and $\sigma_\nu^2 > 0$.

**Remark 1.** Assumption 1 is stronger than the ones often encountered in the literature based on assuming that $N/T \to 0$ (see Westerlund and Breitung, 2013; Westerlund and Larsson, 2009, for some illustrations of the significance of the $N/T \to 0$ assumption), but is standard when testing for a unit root in panels in which $T$ is either fixed or tends to infinity at a
slower rate than \( N \) (see Harris and Tzavalis, 1999; Kruijning, 2009). The assumption that \( v_{i,t} \) is independent across both \( i \) and \( t \), for example, is standard, and is difficult to dispense with unless one is willing to make very specific assumptions regarding the structure of the dependence. One possibility is to assume that the correlation structure is homogenous, as is done in, for example, De Blander and Dhaene (2013). In order to allow more general types of dependencies, however, we require that \( N/T \to 0 \) as \( N, T \to \infty \), which in the current paper is something we want to avoid. It should be pointed out, though, that while \( v_{i,t} \) assumed to be independent, the composite error term \( \varepsilon_{i,t} \) can still be correlated across both \( i \) and \( t \).

Under Assumption 1, \( \varepsilon \sim N(0_{NT \times 1}, \Sigma) \), where

\[
\Sigma = E(\varepsilon \varepsilon') = \sigma_\mu^2 (I_N \otimes I_T I_T') + \sigma_\gamma^2 (I_N I_N' \otimes I_T) + \sigma_\lambda^2 (I_N \otimes I_T) = \Sigma_0 + \sigma_\lambda^2 (I_N I_N' \otimes I_T),
\]

with \( \Sigma_0 = \sigma_\mu^2 [I_N \otimes (I_T + r_T I_T')] \) and \( r_T = \sigma_\mu^2 / \sigma_\nu^2 \). The log-likelihood is therefore given by

\[
\ell(\alpha, \gamma, \sigma_\mu^2, \sigma_\gamma^2, \sigma_\lambda^2) = c - \frac{1}{2} \log(\det \Sigma) - \frac{1}{2} (\Delta y - \alpha y_0 - i_{NT} \gamma)' \Sigma^{-1} (\Delta y - \alpha y_0 - i_{NT} \gamma),
\]

where \( c \) is a generic constant. We will maximize this function in three steps. We begin by concentrating out \( \gamma \), treating \( \alpha \) and the variance parameters as fixed. The resulting maximum likelihood estimator (MLE) \( \hat{\gamma} \) of \( \gamma \) is inserted into \( \ell \), which is then maximized with respect to the variance parameters, again treating \( \alpha \) as fixed. The concentrated log-likelihood function is given by

\[
\ell_c(\alpha) = \ell(\alpha, \hat{\gamma}, \sigma_\mu^2, \sigma_\nu^2, \sigma_\lambda^2),
\]

where \( \sigma_\mu^2, \sigma_\nu^2 \) and \( \sigma_\lambda^2 \) are the second-step MLEs of \( \sigma_\mu^2, \sigma_\nu^2 \) and \( \sigma_\lambda^2 \), respectively. The maximum likelihood under \( H_0 \) is given by \( \ell_c(0) \). The third and final step involves maximizing \( \ell_c \) with respect to \( \alpha \), which yields the maximum log-likelihood under \( H_1 \). Lemma 1, which characterizes \( \ell_c \), is stated in terms of the following projection matrices:

\[
\begin{align*}
P_0 &= I_{NT} - (NT)^{-1} i_{NT} i_{NT}', \\
P_1 &= I_{NT} - T^{-1} (I_N \otimes i_T I_T), \\
P_2 &= I_{NT} - N^{-1} (i_N I_N \otimes I_T).
\end{align*}
\]

where \( P_0 \) wipes out \( \gamma \), while \( P_1 \) (\( P_2 \)) performs the “within” (“between”) transformation that wipes out \( \mu_1, ..., \mu_N \) (\( \lambda_1, ..., \lambda_T \)).

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1One exception is Pesaran (2007), who assumes that \( T/N \to \epsilon > 0 \) as \( N, T \to \infty \) (see also Pesaran et al., 2013). The reason for this rather unusual assumption is that the panel unit root test statistic considered has dependent cross-sections, which invalidates the use of the conventional central limit law argument to asymptotic normality as \( N \to \infty \). The normalization with respect to \( N \), and hence the required expansion rate of \( N \) and \( T \), are therefore not the usual ones.
Lemma 1. Under Assumption 1 the concentrated log-likelihood function is given by
\[ \ell_c(\alpha) \sim c - \frac{NT}{2} \log[\hat{\epsilon}'(P_1 + P_2 - I_{NT})\hat{\epsilon}] - \frac{N}{2} \log[\hat{\epsilon}'(I_N \otimes I_T)\hat{\epsilon}] - \frac{T}{2} \log[\hat{\epsilon}'(I_N \otimes I_T)\hat{\epsilon}], \]
where \( \sim \) signifies asymptotic equivalence and \( \hat{\epsilon} = \hat{\epsilon}(\alpha) = P_0(\Delta y - ay_\cdot) \).

As just mentioned the maximum log-likelihood under \( H_0 \) is simply given by \( \ell_c(0) \). To find the maximum log-likelihood under \( H_1 \) we need to insert \( \hat{\epsilon} = P_0(\Delta y - ay_\cdot) \) into \( \ell_c(\alpha) \) and maximize over \( \alpha \). It seems hard, if not impossible, to do this explicitly, however. Anderson and Hsiao (1981) consider the problem of consistent estimation of \( \alpha \) under the assumption that \( \alpha < 0 \) and \( \sigma^2_\lambda = 0 \) (\( \lambda_1 = \ldots = \lambda_T = 0 \)). Let \( \hat{y}_- = P_0y_- \) and \( \Delta \hat{y} = P_0\Delta y \), such that (3) can be written as \( \Delta \hat{y} = ay_\cdot + \hat{\epsilon} \). The Anderson and Hsiao (1981) covariance (CV) estimator of \( \alpha \) in this “one-way” model (see Baltagi, 2008, Chapter 2) is given by
\[ \hat{\alpha}_{CV1} = \frac{(\Delta \hat{y})' P_1 \hat{y}_-}{\hat{y}_-' P_1 \hat{y}_-}. \]

As Anderson and Hsiao (1981, page 600) point out, under FE \( \hat{\alpha}_{CV1} \) is the MLE. In the RE case, they show that while \( \hat{\alpha}_{CV1} \) is not the exact MLE, the two estimators are asymptotically equivalent as \( T \to \infty \) (with \( N \) fixed or tending to infinity), although they fail to obtain an explicit solution for the MLE. The current estimation problem is more complicated in the sense that \( \sigma^2_\lambda \) need not be zero and \( \alpha = 0 \) under \( H_0 \). It seems reasonable to expect, however, that a generalized “two-way” version of \( \hat{\alpha}_{CV1} \) will inherit some of the properties that apply in the one-way case. A natural suggestion towards this end is
\[ \hat{\alpha}_{CV2} = \frac{(\Delta \hat{y})' P_2 P_1 \hat{y}_-}{\hat{y}_-' P_2 P_1 \hat{y}_-}. \]

By using the same arguments as in Anderson and Hsiao (1981) one can show that \( \hat{\alpha}_{CV2} \) is the MLE under FE (see also Hahn and Moon, 2006). Consider the following bias-corrected (BC) version of \( \hat{\alpha}_{CV2} \):
\[ \hat{\alpha}_{BC} = \hat{\alpha}_{CV2} + \frac{[(\Delta \hat{y})' (I_N \otimes I_T)\hat{y}_-][(\Delta \hat{y})' P_2 P_1 \Delta \hat{y}]}{T[(\Delta \hat{y})' (I_N \otimes I_T)\Delta \hat{y}](\hat{y}_-' P_2 P_1 \hat{y}_-)}. \]

Theorem 1, which is our first main result, characterizes the MLE in the RE case.

**Theorem 1.** Under Assumption 1 and \( H_0 \), as \( T \to \infty \) with \( N \) fixed or \( N \to \infty \), while \( \hat{\alpha}_{CV2} \) is not asymptotically equivalent to the MLE, \( \hat{\alpha}_{BC} \) is.

Hence, in contrast to the stationary unit-specific RE-only case considered by Anderson and Hsiao (1981), in the current setting the CV estimator is not the MLE.
Remark 2. Note how Theorem 1 puts no restrictions on $N$, which can be fixed or tending to infinity. The only requirement in terms of sample size is that $T \to \infty$. This is in agreement with the results of Anderson and Hsiao (1981) for the stationary unit-specific RE-only case where $T \to \infty$ is needed for $\hat{\kappa}_{CV1}$ to be asymptotically equivalent to the MLE.

The asymptotic distribution of $\hat{\kappa}_{BC}$ is provided in Proposition 1. As usual, normality is not necessary for this purpose.

Assumption 2. $\eta_{i,t}$ is iid with $E(\eta_{i,t}) = 0_{3 \times 1}$, $E(\eta_{i,t} \eta'_{i,t}) = \Sigma_\eta = \text{diag}(\sigma^2_\mu, \sigma^2_\lambda, \sigma^2_\nu)$ with $\sigma^2_\mu > 0$ and $\sigma^2_\nu > 0$, and $E(||\eta_{i,t}||^4) < \infty$.

Proposition 1. Under Assumption 2 and $H_0$, as $N, T \to \infty$ with $N/T^5 \to 0$,

$$\sqrt{NT^{3/2}}\hat{\kappa}_{BC} + \sqrt{NT^{-3/2}}6r_\mu^{-2} \to_d \frac{\sqrt{12}}{r_\mu} N(0,1),$$

where $\to_d$ signifies convergence in distribution.

The first thing to note is the rate of consistency, $\sqrt{NT^{3/2}}$, which is higher than the usual panel “superconsistency” rate of $\sqrt{NT}$. The reason for this extraordinarily fast rate of consistency is that under $H_0$, $\mu_i$ generates a linear trend. The trend slopes ($\mu_1, ..., \mu_N$) are mean zero and therefore the mean of $y_{i,t}$ is unaffected. However, the variance increases, leading to a relatively strong unit root signal. This result is closely related to the work of Hahn and Kuersteiner (2002), who consider the asymptotic distribution of $\hat{\kappa}_{CV1}$ in FE case when $\sigma^2_\lambda = 0$ but $\alpha \leq 0$. According to their Theorem 5 the asymptotic distribution (as $N, T \to \infty$ with $N/T \to c \in (0,\infty)$) under $\alpha = 0$ is identical to the one given in our Proposition 1 with $\sigma^2_\mu$ replaced by $\sigma^2_\mu = \lim_{N \to \infty} N^{-1} \sum_{i=1}^N \mu_i^2$. The only difference is the bias, which is $T$ times the bias of $\hat{\kappa}_{BC} - \sqrt{NT^{-3/2}}6r_\mu^{-2}$. Hence, while not completely successful, the proposed correction factor still leads to a substantial reduction in bias. In this sense, $\hat{\kappa}_{BC}$ can be seen as an extension of the BC estimator of Hahn and Kuersteiner (2002), which is only valid when $\alpha < 0$.

Remark 3. The finding that, except for the difference in bias, the asymptotic distribution of $\hat{\kappa}_{BC}$ is the same as that of $\hat{\kappa}_{CV1}$ when $\sigma^2_\lambda = 0$ is in agreement with the results of Hahn and Moon (2006), who consider the asymptotic distribution of $\hat{\kappa}_{CV2}$ under FE and $\alpha < 0$.

---

\footnote{Hahn and Kuersteiner (2002) only consider the case when $\sigma^2_\lambda = 0$. However, as Hahn and Moon (2006) show, their bias correction formula is valid even when $\sigma^2_\lambda > 0$.}
According to their results, the asymptotic distribution of $\hat{\alpha}_{CV2}$ with $\lambda_1, ..., \lambda_T$ unrestricted is the same as that of $\hat{\alpha}_{CV1}$ with $\lambda_1 = ... = \lambda_T = 0$.

The (approximate) LR test statistic for testing $H_0$ versus $H_1$ based on $\hat{\alpha}_{BC}$ is given by

$$LR = -2[\ell_c(0) - \ell_c(\hat{\alpha}_{BC})].$$

**Theorem 2.** Under the conditions of Proposition 1 the following hold as $N, T \to \infty$:

(i) if $N/T^3 \to 0$,

$$LR \to_d \chi^2(1);$$

(ii) if $N/T^3 \to c \in (0, \infty)$,

$$(\sqrt{LR} - \sqrt{3c r^{-3/2}_\mu})^2 \to_d \chi^2(1);$$

(iii) if $N/T^3 \to \infty$ but $N/T^5 \to 0$,

$$\left(\frac{Tr_{\mu}}{12N}\right)^3 (LR - 3NT^{-3}r^{-3}_\mu)^2 \to_d \chi^2(1).$$

The chi-squared limit theory of Theorem 2 is valid for any $N/T$ ratio. The only restriction is that $N/T^5 \to 0$, which has to be considered as very mild. In fact, as far as we are aware this is the most relaxed condition considered so far in the literature (see Breitung and Pesaran, 2008; Baltagi, 2008, Chapter 12, for surveys). Note in particular how Theorem 2 does not require $N/T \to \infty$, and that it applies even in cases when $T/N \to 0$ (see, for example, Pesaran, 2007; Pesaran et al., 2013). As a result, the limit theory is remarkably robust to different sample size constellations of $(N, T)$, suggesting that testing based on Theorem 2 should show little size distortion, a result that is verified below using Monte Carlo simulation.

**Remark 4.** Note how the results reported in Theorem 2 only depend on $\sigma^2_v$ and $\sigma^2_\mu$ (via $r_\mu$) and not on $\sigma^2_\lambda$. This is consistent with the finding of Hahn and Moon (2006) that the bias of $\hat{\alpha}_{CV2}$ in the stationary FE case is due to the presence of $\mu_i$ in (1), and not the presence of $\lambda_t$ (see also Moon at el., 2013).

**Remark 5.** It can be shown that Theorem 2 holds also in the FE case, provided that $\sigma^2_\mu$ is replaced by $\sigma^2_\mu$ (see the discussion following Proposition 1). The proposed LR test statistic is therefore robust against misspecification of the effects.
A small-scale Monte Carlo study was carried out to assess the accuracy of the theoretical predictions given in Theorem 2. The data generating process used for this purpose is a restricted version of the one that applies under Assumption 1, and sets $\Sigma_\eta = I_3$. Our main interest here is how the performance changes as a function of $N/T$. We consider both “micro panels” with $N = 1,000$ and $T$ between 5 and 20, and “macro panels” with $T = 100$ and $N$ between 5 and 20. Three versions of the LR test statistic are considered, which differ only in how $LR$ is normalized to enable chi-squared inference, as indicated in Theorem 2 (i)–(iii). Let us denote these as $LR_1 = LR$, $LR_{02} = (\sqrt{LR} - \sqrt{3} r_\mu^{-3/2})^2$ and $LR_{03} = (Tr_\mu)^3 (LR - 3NT^{-3} r_\mu^{-3})^2 / (12N)$, where $c = N/T^3$. Of course, in practice, $r_\mu$ is never unknown, but we can estimate it as $\hat{r}_\mu = \hat{\sigma}_\nu^2 / \hat{\sigma}_\mu^2$, where $\hat{\sigma}_\nu^2 = (NT)^{-1} \Delta y' (P_1 + P_2 - I_{NT}) \Delta y$ and $\hat{\sigma}_\mu^2 = T^{-1} [(NT)^{-1} \Delta y' (I_N \otimes IT') \Delta y - \hat{\sigma}_\nu^2]$ (see Proof of Lemma 1 for a derivation of these estimators). The resulting feasible versions of $LR_{02}$ and $LR_{03}$ are henceforth denoted $LR_2$ and $LR_3$, respectively. We focus on the size and power at the 5% level, although we also report some results on the relative bias and root mean squared error (RMSE) of $\hat{\alpha}_{BC}$ when compared to $\hat{\alpha}_{CV2}$. The number of replications is set to 5,000. All computational work was done in GAUSS.

The bias and RMSE results reported in Table 1 show that the performance of $\hat{\alpha}_{BC}$ is uniformly better than that of $\hat{\alpha}_{CV2}$. The biggest difference occurs when $T$ is relatively small; however, there are some important differences also when $T$ is large. As expected, the results reported in Table 2 reveal that the size properties of $LR_1$ and $LR_3$ are rather complementary; when $N >> T$, $LR_3$ works best, whereas when $T >> N$, it is the other way around. The most striking feature of Table 1 is, however, the good performance of $LR_2$. Of course, size accuracy is not perfect and some distortions seem to remain. Given the range of values for $N$ and $T$ considered, however, the distortions are still acceptable. Note in particular how $LR_2$ seems to maintain good size accuracy when $T$ is as small as 10, which is uncommon even for tests that only require $N \rightarrow \infty$ with $T$ held fixed (see, for example, Hadri and Larsson, 2005). As expected, estimation of $r_\mu$ is generally not detrimental for test performance, at least not for $LR_2$. The only exception is when $N = 1,000$ and $T = 5$, in which case the test based on estimating $r_\mu$ is quite oversized. However, the distortions vanish very quickly as $T$ increases.

3 The LR test statistic based on the exact (numerical) MLE was also simulated. In this case, we used OPTMUM, which was implemented in its default setting (using the BFGS algorithm). The results were, however, almost identical to those obtained for $\hat{\alpha}_{BC}$. We therefore omit them.
3 Conclusion

This paper develops a simple test for a unit root in dynamic panel data models with random effects. The new test statistic, which is based on the LR principle, is shown to have a standard chi-squared distribution as \( N, T \to \infty \) with \( N/T^5 \to \infty \). As a result, the limit theory is very robust to different sample size constellations of \((N, T)\), and simulations support the resulting intuition that testing based on this theory should show little size distortion.
References


Appendix

Proof of Lemma 1.

The first-step maximizer of $\gamma$ is easily seen to be given by

$$\hat{\gamma} = (NT)^{-1}i_{NT'}(\Delta y - ay_-),$$

leading to the following concentrated log-likelihood:

$$\ell(a, \hat{\gamma}, \sigma^2_{\mu}, \sigma^2_{v}, \sigma^2_{\lambda}) = c - \frac{1}{2} \log(\det \Sigma) - \frac{1}{2} \hat{\epsilon}' \Sigma^{-1} \hat{\epsilon},$$ \hspace{1cm} (6)

where $\hat{\epsilon} = \Delta y - ay_-$ and $P_0 = I_{NT} - (NT)^{-1}i_{NT}i_{NT'}$.

Next, we maximize (6) with respect to $(\sigma^2_{\mu}, \sigma^2_{\lambda}, \sigma^2_{v})$. Let $r_k = \sigma^2_k/\sigma^2_v$ for any $k \in \{\mu, \lambda\}$.

For a scalar $a$ and a vector $v$ we have the identities $(I + avv')^{-1} = I - (avv')/(1 + av'v)$ and $\det(I + avv') = 1 + av'v$, which yield

$$(I_T + r_\mu i_T i_T')^{-1} = I_T - r_\mu i_T i_T'(1 + r_\mu T)^{-1} = I_T - \phi_\mu i_T i_T',$$

where $\phi_\mu = \sigma^2_{\mu}/\omega^2_\mu$ and $\omega^2_\mu = \sigma^2_{\mu} + N\sigma^2_{\lambda}$. Let us similarly define $\phi_{\lambda} = \sigma^2_{\lambda}/\omega^2_{\lambda}$, where $\omega^2_{\lambda} = \sigma^2_{\lambda} + N\sigma^2_{\lambda}$. By using this, $(A + BC)^{-1} = A^{-1} - A^{-1}B(I + CA^{-1}B)^{-1}CA^{-1}$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, we obtain

$$\Sigma^{-1} = \left[\Sigma_0 + \sigma^2_{\lambda}(i_N \otimes I_T)(i_N \otimes I_T)\right]^{-1}$$

$$= \Sigma_0^{-1} - \Sigma_0^{-1}\sigma^2_{\lambda}(i_N \otimes I_T)(I_T + \sigma^2_{\lambda}(i_N \otimes I_T)'\Sigma_0^{-1}(i_N \otimes I_T))^{-1}(i_N \otimes I_T)\Sigma_0^{-1}$$

$$= \Sigma_0^{-1} - \Sigma_0^{-1}\sigma^2_{\lambda}(i_N \otimes I_T)(I_T + Nr_{\lambda}(I_T - \phi_\mu i_T i_T'))^{-1}(i_N \otimes I_T)\Sigma_0^{-1}$$

$$= \Sigma_0^{-1} - \sigma^2_{\lambda}i_T\Sigma_0^{-1}(i_N \otimes I_T)(I_T + \frac{N\phi_\mu \phi_{\lambda}}{1 - NT\phi_\mu \phi_{\lambda}}i_T i_T') (i_N \otimes I_T)\Sigma_0^{-1}$$

$$= \Sigma_0^{-1} - \sigma^2_{\lambda}i_T\Sigma_0^{-1} \left[ i_N i_T' \otimes \left(I_T + \frac{N\phi_\mu \phi_{\lambda}}{1 - NT\phi_\mu \phi_{\lambda}}i_T i_T' \right) \right] \Sigma_0^{-1}$$

$$= \sigma^2_{\lambda}^{-2} [i_N \otimes (I_T - \phi_\mu i_T i_T')]$$

$$- \sigma^2_{\lambda}^{-1} \phi_{\lambda} \left[i_N i_T' \otimes (I_T - \phi_\mu i_T i_T') \left(I_T + \frac{N\phi_\mu \phi_{\lambda}}{1 - NT\phi_\mu \phi_{\lambda}}i_T i_T' \right) \right]$$

$$= \sigma^2_{\lambda}^{-2} [i_N \otimes (I_T - \phi_\mu i_T i_T')] - \sigma^2_{\lambda}^{-1} \phi_{\lambda} \left[i_N i_T' \otimes \left(I_T - \phi_\mu \frac{\omega^2_{\mu} + \omega^2_{\lambda}}{\omega^2_{\mu} + \omega^2_{\lambda} - \sigma^2_{\lambda} i_T i_T'} \right) \right].$$

Moreover,

$$\det(I_T + r_\mu i_T i_T') = 1 + r_\mu T \frac{\sigma^2_{\lambda} + T\sigma^2_{\mu}}{\sigma^2_{\lambda}} \frac{r_\mu}{\phi_{\mu}}.$$
We also have \(\det(\Sigma_0 + BC) = \det \Sigma_0 \det(I + C\Sigma_0^{-1}B)\) and \(\det(I_N \otimes \Sigma_0) = (\det \Sigma_0)^N\), suggesting that

\[
\det \Sigma \quad = \quad \det[\Sigma_0 + \sigma^2 \mathbf{T}(\mathbf{T} \otimes \mathbf{T})(\mathbf{T} \otimes \mathbf{T})']
\]

\[
\quad = \quad \det \Sigma_0 \det[I_T + \sigma^2 \mathbf{T}(\mathbf{T} \otimes \mathbf{T})^\prime \Sigma_0^{-1}(\mathbf{T} \otimes \mathbf{T})]
\]

\[
\quad = \quad \sigma^2 N T \left( \frac{r_\mu}{\phi_\mu} \right)^N \det[I_T + N r_\lambda(I_T - \phi_\mu T T_T')]
\]

\[
\quad = \quad \sigma^2 N \mathbf{T} \mathbf{T}^\prime (1 + N r_\lambda)^T \det(I_T - N \phi_\mu \phi_\lambda T T_T')
\]

\[
\quad = \quad \sigma^2 \mathbf{T} \mathbf{T}^\prime \mathbf{T}^\prime \Phi (1 - NT \phi_\lambda)
\]

\[
\quad = \quad \sigma^2 \mathbf{T} \mathbf{T}^\prime \Phi (1 - NT \phi_\lambda).
\]

Insertion into (6) now yields

\[
\ell(\alpha, \gamma, \sigma^2, \sigma^2, \sigma^2) = c - \frac{1}{2} \log(\det \Sigma) - \frac{1}{2} \xi^2 \Sigma^{-1} \xi
\]

\[
\quad = \quad c - \frac{NT - N - T}{2} \log(\sigma^2_v) - \frac{N}{2} \log(\omega^2_\mu) - \frac{T}{2} \log(\omega^2_\lambda) - \frac{1}{2} \log(1 - NT \phi_\mu \phi_\lambda)
\]

\[
\quad = \quad c - \frac{1}{2} \xi^2 \left[ I_N \otimes (I_T - \phi_\mu T T_T') - \Phi \left( \mathbf{T} \otimes \left( I_T - \phi_\mu \Phi \omega^2_\mu \omega^2_\lambda - \sigma^2_v \phi_\lambda T T_T' \right) \right) \right] \xi
\]

\[
\quad = \quad c - \frac{1}{2} \xi^2 \left[ I_N \otimes (I_T - \phi_\mu T T_T') \right] \xi
\]

\[
\quad = \quad c - \frac{1}{2} \xi^2 \left[ (I_N \otimes (I_T - \phi_\mu T T_T')) - \Phi \left( \mathbf{T} \otimes \left( I_T - \phi_\mu \Phi \omega^2_\mu \omega^2_\lambda - \sigma^2_v \phi_\lambda T T_T' \right) \right) \right] \xi,
\]

(7)

where the last equality holds, because \((\mathbf{T} \otimes \mathbf{T} T_T')\xi = (\mathbf{T} \otimes \mathbf{T} T_T') P_0 \xi = 0\).

Define

\[
\xi_0 \quad = \quad \xi^2 \xi,
\]

\[
\xi_1 \quad = \quad \Phi \left( \mathbf{T} \otimes (I_T - \phi_\mu T T_T') \right) \xi,
\]

\[
\xi_2 \quad = \quad \Phi \left( \mathbf{T} \otimes (I_T - \phi_\mu T T_T') \right) \xi.
\]

We now use

\[
1 - NT \phi_\mu \phi_\lambda = 1 - \frac{NT \sigma^2_\mu \sigma^2_\lambda}{\sigma^2_\mu \omega^2_\lambda} = \frac{\omega^2_\mu}{\omega^2_\lambda} \frac{\omega^2_\mu + \omega^2_\lambda - \sigma^2_v}{\omega^2_\mu \omega^2_\lambda},
\]

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to rewrite (7) as

\[ \ell(\alpha, \gamma, \sigma^2_\mu, \sigma^2_\nu, \sigma^2_\lambda) \]
\[ = c - \frac{NT - N - T}{2} \log(\sigma^2_\nu) - \frac{N}{2} \log(\omega^2_\mu) - \frac{T}{2} \log(\omega^2_\lambda) - \frac{1}{2} \log(1 - NT\phi_\mu\phi_\lambda) \]
\[ - \frac{1}{2\sigma^2_\nu} (\xi_0 - T\phi_\mu\xi_1 - N\phi_\lambda\xi_2) \]
\[ = c - \frac{(N - 1)(T - 1)}{2} \log(\sigma^2_\nu) - \frac{(N - 1)}{2} \log(\omega^2_\mu) - \frac{(T - 1)}{2} \log(\omega^2_\lambda) \]
\[ - \frac{1}{2} \log(\omega^2_\mu + \omega^2_\lambda - \sigma^2_\nu) - \frac{1}{2\sigma^2_\nu} (\xi_0 - T\phi_\mu\xi_1 - N\phi_\lambda\xi_2). \] (8)

Hence,

\[ \frac{\partial \ell}{\partial (\sigma^2_\nu)} = \frac{(N - 1)(T - 1)}{2\sigma^2_\nu} - \frac{(N - 1)}{2\omega^2_\mu} - \frac{(T - 1)}{2\omega^2_\lambda} - \frac{1}{2(\omega^2_\mu + \omega^2_\lambda - \sigma^2_\nu)} \]
\[ + \frac{1}{2\sigma^2_\nu} (\xi_0 - T\phi_\mu\xi_1 - N\phi_\lambda\xi_2) - \frac{1}{2\sigma^2_\nu} \left( \frac{T\phi_\mu}{\omega^2_\mu} \xi_1 + \frac{N\phi_\lambda}{\omega^2_\lambda} \xi_2 \right), \]
\[ \frac{\partial \ell}{\partial (\sigma^2_\mu)} = \frac{(N - 1)T}{2\omega^2_\mu} - \frac{T}{2(\omega^2_\mu + \omega^2_\lambda - \sigma^2_\nu)} + \frac{T}{2\omega^2_\lambda} \xi_1, \]
and
\[ \frac{\partial \ell}{\partial (\sigma^2_\lambda)} = -\frac{N(T - 1)}{2\omega^2_\lambda} - \frac{N}{2(\omega^2_\mu + \omega^2_\lambda - \sigma^2_\nu)} + \frac{N}{2\omega^2_\lambda} \xi_2. \]

We therefore have the following first-order conditions:

\[ 0 = \frac{(N - 1)(T - 1)}{2\sigma^2_\nu} - \frac{(N - 1)}{2\omega^2_\mu} - \frac{(T - 1)}{2\omega^2_\lambda} - \frac{1}{2(\omega^2_\mu + \omega^2_\lambda - \sigma^2_\nu)} \]
\[ + \frac{1}{2\sigma^2_\nu} (\xi_0 - \omega^2_\mu - \sigma^2_\nu \xi_1 - \omega^2_\lambda - \sigma^2_\nu \xi_2) - \frac{1}{2\sigma^2_\nu} \left( \frac{\omega^2_\mu - \sigma^2_\nu}{\omega^4_\mu} \xi_1 + \frac{\omega^2_\lambda - \sigma^2_\nu}{\omega^4_\lambda} \xi_2 \right) \]
\[ = \frac{(N - 1)(T - 1)}{2\sigma^2_\nu} - \frac{(N - 1)}{2\omega^2_\mu} - \frac{(T - 1)}{2\omega^2_\lambda} - \frac{1}{2(\omega^2_\mu + \omega^2_\lambda - \sigma^2_\nu)} \]
\[ + \frac{1}{2\sigma^2_\nu} (\xi_0 - \xi_1 - \xi_2) + \frac{1}{2} \left( \frac{\xi_1}{\omega^2_\mu} + \frac{\xi_2}{\omega^2_\lambda} \right), \] (9)

\[ 0 = -\frac{(N - 1)T}{2\omega^2_\mu} - \frac{T}{2(\omega^2_\mu + \omega^2_\lambda - \sigma^2_\nu)} + \frac{T}{2\omega^2_\lambda} \xi_1, \] (10)
and

\[ 0 = -\frac{N(T - 1)}{2\omega^2_\lambda} - \frac{N}{2(\omega^2_\mu + \omega^2_\lambda - \sigma^2_\nu)} + \frac{N}{2\omega^2_\lambda} \xi_2. \] (11)
The two last conditions may be rewritten as

\[
((N - 1)\hat{\omega}_\mu^2 - \xi_1)(\hat{\omega}_\mu^2 + \hat{\omega}_\lambda^2 - \hat{\sigma}_v^2) + \hat{\omega}_\mu^4 = 0,
\]

\[
((T - 1)\hat{\omega}_\lambda^2 - \xi_2)(\hat{\omega}_\mu^2 + \hat{\omega}_\lambda^2 - \hat{\sigma}_v^2) + \hat{\omega}_\lambda^4 = 0,
\]

which are in turn asymptotically equivalent to

\[
(NT\hat{\omega}_\mu^2 - \xi_1)(\hat{\omega}_\mu^2 + \hat{\omega}_\lambda^2 - \hat{\sigma}_v^2) = 0,
\]

\[
(T\hat{\omega}_\lambda^2 - \xi_2)(\hat{\omega}_\mu^2 + \hat{\omega}_\lambda^2 - \hat{\sigma}_v^2) = 0.
\]

Hence, provided that \(\hat{\omega}_\mu^2 + \hat{\omega}_\lambda^2 - \hat{\sigma}_v^2 \neq 0\), we have

\[
\hat{\omega}_\mu^2 \sim N^{-1}\xi_1, \quad (12)
\]

\[
\hat{\omega}_\lambda^2 \sim T^{-1}\xi_2. \quad (13)
\]

Similarly, (9) may be rewritten as

\[
(NT - 1)(T - 1)\hat{\omega}_\mu^4\hat{\omega}_\lambda^4\hat{\sigma}_v^4((\hat{\omega}_\mu^2 + \hat{\omega}_\lambda^2 - \hat{\sigma}_v^2) + (N - 1)\hat{\omega}_\mu^2\hat{\omega}_\lambda^4\hat{\sigma}_v^4(\hat{\omega}_\mu^2 + \hat{\omega}_\lambda^2 - \hat{\sigma}_v^2) + (T - 1)\hat{\omega}_\mu^4\hat{\omega}_\lambda^2\hat{\sigma}_v^4(\hat{\omega}_\mu^2 + \hat{\omega}_\lambda^2 - \hat{\sigma}_v^2) + \hat{\omega}_\mu^4\hat{\omega}_\lambda^2\hat{\sigma}_v^4(\hat{\omega}_\mu^2 + \hat{\omega}_\lambda^2 - \hat{\sigma}_v^2)
\]

which is asymptotically equivalent to

\[
NT\hat{\omega}_\mu^4\hat{\omega}_\lambda^4\hat{\sigma}_v^4 + (N\hat{\omega}_\mu^2\hat{\omega}_\lambda^4 + T\hat{\omega}_\mu^4\hat{\omega}_\lambda^2)\hat{\sigma}_v^4 = \hat{\omega}_\mu^4\hat{\omega}_\lambda^4(\xi_0 - \xi_1 - \xi_2) + (\xi_1\hat{\omega}_\mu^4 + \xi_2\hat{\omega}_\mu^4)\hat{\sigma}_v^4.
\]

Inserting (12) and (13) this expression simplifies into

\[
NT\hat{\sigma}_v^2 \sim (\xi_0 - \xi_1 - \xi_2), \quad (14)
\]

Insertion of (12)–(14) into (8) now yields

\[
\ell_c(\alpha) = \ell(\alpha, \hat{\gamma}, \hat{\omega}_\sigma^2, \hat{\omega}_v^2, \hat{\sigma}_v^2) \\
\sim c - \frac{NT}{2} \log((NT)^{-1}(\xi_0 - \xi_1 - \xi_2)) - \frac{N}{2} \log(N^{-1}\xi_1) - \frac{T}{2} \log(T^{-1}\xi_2) \\
\sim c - \frac{NT}{2} \log(\xi_0 - \xi_1 - \xi_2) - \frac{N}{2} \log(T\xi_1) - \frac{T}{2} \log(N\xi_2), \quad (15)
\]

as was to be shown.
Before we come to the proof of Theorems 1 and 2 we need the following lemma, which is stated in terms of the following quantities:

\[
\begin{align*}
  a_{10} &= (\Delta \hat{y})' (I_N \otimes t_T t'_T) \Delta \hat{y}, \\
  a_{11} &= (\Delta \hat{y})' (I_N \otimes t_T t'_T) \hat{y}_-, \\
  a_{12} &= (\hat{y}_-)' (I_N \otimes t_T t'_T) \hat{y}_-, \\
  a_{20} &= (\Delta \hat{y})' (I_N' \otimes I_T) \Delta \hat{y}, \\
  a_{21} &= (\Delta \hat{y})' (I_N' \otimes I_T) \hat{y}_-, \\
  a_{22} &= (\hat{y}_-)' (I_N' \otimes I_T) \hat{y}_-, \\
  b_0 &= (\Delta \hat{y})' P_2 P_1 P_2 \Delta \hat{y}, \\
  b_1 &= (\Delta \hat{y})' P_2 P_1 P_2 \hat{y}_-, \\
  b_2 &= (\hat{y}_-)' P_2 P_1 P_2 \hat{y}_-.
\end{align*}
\]

Lemma A.1. Under \( H_0 \),

\[
E(a_{10}) = (N - 1)T(\sigma^2_v + T\sigma^2_\mu),
\]

\[
E(a_{11}) = \frac{1}{2}(N - 1)T(T - 1)(\sigma^2_v + T\sigma^2_\mu),
\]

\[
E(a_{12}) = \sigma^2_v \left( \frac{1}{6}NT(T - 1)(2T - 1) - \frac{1}{4}T(T - 1)^2 \right) + \frac{1}{4}\sigma^2_\mu(N - 1)T^2(T - 1)^2
\]

\[
+ \frac{1}{12}\sigma^2_\lambda NT(T^2 - 1),
\]

\[
E(a_{20}) = N(T - 1)(\sigma^2_v + N\sigma^2_\lambda),
\]

\[
E(a_{21}) = -\frac{1}{2}N(T - 1)(\sigma^2_v + N\sigma^2_\lambda),
\]

\[
E(a_{22}) = \frac{1}{6}N(\sigma^2_v + N\sigma^2_\lambda)(T - 1)(2T + 5),
\]

\[
E(b_0) = \sigma^2_v(N - 1)(T - 1),
\]

\[
E(b_1) = -\frac{1}{2}(N - 1)(T - 1)\sigma^2_v,
\]

\[
E(b_2) = \frac{1}{12}\sigma^2_\mu(N - 1)T(T^2 - 1) - \frac{1}{6}\sigma^2_v(N - 1)(T - 1)(2T - 7).
\]

Proof of Lemma A.1.

Because \( P_0 = I_{NT}(NT)^{-1}I_{NT} t'_N \) and \( \Delta \hat{y} = P_0 \hat{\varepsilon} \), we have

\[
E(a_{10}) = \text{tr}[(I_N \otimes t_T t'_T)P_0 E(\varepsilon \varepsilon')P_0] = \text{tr}[P_0(I_N \otimes t_T t'_T)P_0 \Sigma],
\]
where
\[
P_0(I_N \otimes \iota_T l'_T) = [I_{NT} - (NT)^{-1}i_{NT} l'_N](I_N \otimes \iota_T l'_T)
\]
\[
= (I_N \otimes \iota_T l'_T) - (NT)^{-1}i_{NT} l'_N(I_N \otimes \iota_T l'_T)
\]
\[
= (I_N \otimes \iota_T l'_T) - N^{-1}i_{NT} l'_N \otimes \iota_T l'_T
\]
\[
= (I_N - N^{-1}i_{NT} l'_N) \otimes \iota_T l'_T. \tag{16}
\]

By using this and \((I_N - N^{-1}i_{NT} l'_N)\iota_N = 0\), we obtain
\[
P_0(I_N \otimes \iota_T l'_T)P_0 = (I_N - N^{-1}i_{NT} l'_N) \otimes \iota_T l'_T. \tag{17}
\]

Hence, in view of (4),
\[
E(a_{10}) = \sigma_1^2 \text{tr}[(I_N - N^{-1}i_{NT} l'_N) \otimes \iota_T l'_T](I_N \otimes \iota_T l'_T)
\]
\[
+ \sigma_2^2 \text{tr}[(I_N - N^{-1}i_{NT} l'_N) \otimes \iota_T l'_T](i_{NT} \otimes \iota_T)
\]
\[
+ \sigma_3^2 \text{tr}[(I_N - N^{-1}i_{NT} l'_N) \otimes \iota_T l'_T]
\]
\[
= (N - 1)T(\sigma_1^2 + T\sigma_2^2) = O(NT^2). \tag{18}
\]

Consider \(E(a_{11})\). Writing \(M = L(I_T - L)^{-1}\), where \(L\) is the \(T \times T\) lag matrix
\[
L = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & 0
\end{bmatrix},
\]
we have
\[
\hat{y}_- = (I_N \otimes L)\hat{y} = (I_N \otimes M)\Delta\hat{y}. \tag{19}
\]

By using this and (16),
\[
E(a_{11}) = E(\Delta\hat{y}^r(I_N \otimes \iota_T l'_T)(I_N \otimes M)\Delta\hat{y})
\]
\[
= \text{tr}(E[(I_N \otimes \iota_T l'_T)(I_N \otimes M)\Delta\hat{y}(\Delta\hat{y})^r])
\]
\[
= \text{tr}[[I_N \otimes \iota_T l'_T](I_N \otimes M)P_0\Sigma P_0]
\]
\[
= \text{tr}[P_0(I_N \otimes \iota_T l'_T)(I_N \otimes M)P_0\Sigma]
\]
\[
= \text{tr}[[I_N - N^{-1}i_{NT} l'_N) \otimes \iota_T l'_T](I_N \otimes M)P_0\Sigma]
\]
\[
= \text{tr}([I_N - N^{-1}i_{NT} l'_N) \otimes \iota_T l'_T](I_N \otimes M)\Sigma)
\]
\[
= (NT)^{-1}i_{NT} l'_N[I_N - N^{-1}i_{NT} l'_N) \otimes \iota_T l'_T](I_N \otimes M)\Sigma i_{NT},
\]

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where the second term on the right-hand side is zero, because \((I_N - N^{-1}t_N')t_N = 0\). Hence, via (4),

\[
E(a_{11}) = \sigma^2_v \text{tr}((I_N - N^{-1}t_N') \otimes t_T') (I_N \otimes M) (I_N \otimes t_T') \\
+ \sigma^2_v \text{tr}((I_N - N^{-1}t_N') \otimes t_T') (I_N \otimes (I_N t_N' \otimes I_T)) \\
+ \sigma^2_v \text{tr}((I_N - N^{-1}t_N') \otimes t_T') (I_N \otimes M) \\
= (N - 1)(\sigma^2_v + T\sigma^2_p) t_T M_T. \tag{20}
\]

Note that

\[
M = L(I_T - L)^{-1} = \sum_{j=1}^{T-1} L_j, \tag{21}
\]

suggesting that, since \(t_T' L_j = T - j\), we have

\[
t_T' M_T = \sum_{j=1}^{T} (T - j) = \frac{1}{2} T(T - 1). \tag{22}
\]

Insertion into (20) gives

\[
E(a_{11}) = \frac{1}{2} (N - 1) T(T - 1)(\sigma^2_v + T\sigma^2_p). \tag{23}
\]

For \(E(a_{12})\), making use of (21), we obtain

\[
E(a_{12}) = E[(\Delta y)' (I_N \otimes M') (I_N \otimes t_T') (I_N \otimes M) \Delta y] \\
= \text{tr}[P_0 (I_N \otimes M') (I_N \otimes t_T') (I_N \otimes M) P_0 \Sigma] \\
= \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}[P_0 (I_N \otimes L_j' t_T' L_k^T) P_0 \Sigma]. \tag{24}
\]

From (4),

\[
\text{tr}[P_0 (I_N \otimes L_j' t_T' L_k^T) P_0 \Sigma] = \sigma^2_v \tau_1 + \sigma^2_p \tau_2 + \sigma^2_{\lambda} \tau_3, \tag{25}
\]

where

\[
\tau_1 = \text{tr}[P_0 (I_N \otimes L_j' t_T' L_k^T) P_0] = \text{tr}[P_0 (I_N \otimes L_j' t_T' L_k^T)] \\
\tau_2 = \text{tr}[(I_N \otimes L_j' t_T' L_k^T) P_0 (I_N \otimes t_T') P_0], \\
\tau_3 = \text{tr}[(I_N \otimes L_j' t_T' L_k^T) P_0 (t_N t_N' \otimes I_T) P_0].
\]

Because \(t_T' L_j^T = (i_{T-j}, 0_{n-T})\), we get

\[
\tau_1 = N t_T L_j^T L_j' - (NT)^{-1} t_N (I_N \otimes L_j' t_T' L_k^T) t_N \\
= N(T - j \lor k) - T^{-1}(T - j)(T - k),
\]

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and by further use of (17),

\[ \tau_2 = \text{tr}[(I_N \otimes L^j l_TT^k)((I_N - N^{-1}l_N) \otimes l_Tl_T)] \]
\[ = (N - 1)l_T^2L^j l_TT^k = (N - 1)(T - j)(T - k). \]

Analogous to (17), we have

\[ P_0(l_Nl_N' \otimes I_T)P_0 = l_Nl_N' \otimes (I_N - T^{-1}l_Tl_T'), \]

suggesting that

\[ \tau_3 = \text{tr}[(I_N \otimes L^j l_TT^k)(l_Nl_N' \otimes (I_N - T^{-1}l_Tl_T'))]] \]
\[ = N(l_T^2l_T - T^{-1}l_T^2l_Tl_T) \]
\[ = N[(T - j \lor k) - T^{-1}(T - j)(T - k)]. \]

Hence, because

\[ \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} (T - j \lor k) = \sum_{j=1}^{T-1} (T - j) + 2 \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} (T - j) = \frac{1}{6} T(T - 1)(2T - 1), \] (26)

we get, via (24) and (25),

\[ E(a_{12}) = \sigma_v^2 \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} [N(T - j \lor k) - T^{-1}(T - j)(T - k)] \]
\[ + \sigma_v^2 \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} (N - 1)(T - j)(T - k) + \sigma^2 \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} N[(T - j \lor k) - T^{-1}(T - j)(T - k)] \]
\[ = \sigma_v^2 \left( \frac{1}{6} NT(T - 1)(2T - 1) - \frac{1}{4} T(T - 1)^2 \right) + \frac{1}{4} \sigma^2 \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} (N - 1)T^2(T - 1)^2 \]
\[ + \frac{1}{12} \sigma^2 \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} (T^2 - 1). \] (27)

\[ E(a_{20}) \text{ follows by symmetry;} \]
\[ E(a_{20}) = N(T - 1)(\sigma_v^2 + N\sigma^2). \] (28)

For \(E(a_{21})\),

\[ E(a_{21}) = E[(\Delta \tilde{y})' (l_Nl_N' \otimes I_T)(I_N \otimes M)\Delta \tilde{y}] = \text{tr}[P_0(l_Nl_N' \otimes I_T)(I_N \otimes M)P_0\Sigma], \]

which, in view of \(P_0(l_Nl_N' \otimes I_T) = l_Nl_N' \otimes (I_T - T^{-1}l_Tl_T')\), becomes

\[ E(a_{21}) = \text{tr}[(I_Nl_N' \otimes (I_T - T^{-1}l_Tl_T'))(I_N \otimes M)P_0\Sigma] \]
\[ = \text{tr}[(I_Nl_N' \otimes (I_T - T^{-1}l_Tl_T'))(I_N \otimes M)\Sigma_{lNT}] \]
\[ = \text{tr}[(I_Nl_N' \otimes (I_T - T^{-1}l_Tl_T'))(I_N \otimes M)\Sigma_{lNT}], \]

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where the second term is zero. Use of (4), (21) and (22) now yield

\[
E(a_{21}) = \sigma^2_\nu \text{tr}[(i_{NT} l'_N \otimes (I_T - T^{-1} T') \otimes (I_N \otimes M) (I_N \otimes i_T l'_T)] + \sigma^2_\nu \text{tr}[(i_{NT} l'_N \otimes (I_T - T^{-1} T') \otimes (I_N \otimes M) (i_N l'_N \otimes I_T)] + \sigma^2_\nu \text{tr}[(i_{NT} l'_N \otimes (I_T - T^{-1} T') \otimes (I_N \otimes M)]
\]

\[
= \sigma_\nu^2 N l'_T (I_T - T^{-1} T') M_{II} + \sigma_\nu^2 N [\text{tr}(M) - T^{-1} l'_T M_{II}] + \sigma_\nu^2 N [\text{tr}(M) - T^{-1} l'_T M_{II}]
\]

\[
= -\frac{1}{2} N(T - 1)(\sigma_\nu^2 + N \sigma_\lambda^2). \tag{29}
\]

Consider \(E(a_{22})\). From (21),

\[
E(a_{22}) = E[(\Delta \hat{\eta})(I_N \otimes M')(i_{NT} l'_N \otimes I_T)(I_N \otimes M) \Delta \hat{\eta}]
\]

\[
= \text{tr}[P_0 (I_N \otimes M')(i_{NT} l'_N \otimes I_T)(I_N \otimes M) P_0 \Sigma]
\]

\[
= \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}[P_0 (i_{NT} l'_N \otimes L^j L^k) P_0 \Sigma],
\]

where

\[
P_0 (i_{NT} l'_N \otimes L^j L^k) P_0
\]

\[
= (i_{NT} l'_N \otimes L^j L^k) - (NT)^{-1} i_{NT} i_{NT} (i_{NT} l'_N \otimes L^j L^k) - (NT)^{-1} (i_{NT} l'_N \otimes L^j L^k) i_{NT} i_{NT}
\]

\[
+ (NT)^{-2} (i_{NT} l'_N \otimes L^j L^k) i_{NT} i_{NT}
\]

\[
= (i_{NT} l'_N \otimes L^j L^k) - T^{-1} (i_{NT} l'_N \otimes i_T l'_T L^j L^k) - T^{-1} (i_{NT} l'_N \otimes L^j L^k i_T l'_T)
\]

\[
+ T^{-2} (i_{NT} l'_N \otimes i_T l'_T L^j L^k i_T l'_T).
\]

Let \(1(A) = 1\) if \(A\) is true and 0 otherwise. By (4),

\[
\text{tr}[P_0 (i_{NT} l'_N \otimes L^j L^k) P_0 \Sigma] = \sigma_\nu^2 \kappa_1 + \sigma_\nu^2 \kappa_2 + \sigma_\lambda^2 \kappa_3,
\]

where

\[
\kappa_1 = \text{tr}[(i_{NT} l'_N \otimes L^j L^k)] - T^{-1} \text{tr}[(i_{NT} l'_N \otimes i_T l'_T L^j L^k)]
\]

\[
= N[1(j = k) - T^{-1}(T - j \lor k)],
\]

\[
\kappa_2 = \text{tr}[(i_{NT} l'_N \otimes L^j L^k) (I_N \otimes i_T l'_T)] - T^{-1} \text{tr}[(i_{NT} l'_N \otimes i_T l'_T L^j L^k) (I_N \otimes i_T l'_T)] = 0,
\]

\[
\kappa_3 = \text{tr}[(i_{NT} l'_N \otimes L^j L^k)(i_{NT} l'_N \otimes I_T)] - T^{-1} \text{tr}[(i_{NT} l'_N \otimes i_T l'_T L^j L^k)(i_{NT} l'_N \otimes I_T)]
\]

\[
= N^2[1(j = k) - T^{-1}(T - j \lor k)],
\]
and so, via (26),

\[
E(a_{22}) = N(\sigma_v^2 + N\sigma_\lambda^2) \left( T - 1 + T^{-1} \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} (T - j \vee k) \right)
\]

\[
= N(\sigma_v^2 + N\sigma_\lambda^2) \left( T - 1 + \frac{1}{6} (T - 1) (2T - 1) \right)
\]

\[
= \frac{1}{6} N(\sigma_v^2 + N\sigma_\lambda^2) (T - 1)(2T + 5).
\]

(30)

Next, consider \( E(b_0) \). By using (4), (16) and \( P_0(i_{NI'_N} \otimes I_T) = i_{NI'_N} \otimes (I_T - T^{-1}I_T I_T') \), it is clear that

\[
E(b_0) = E[(\Delta\hat{y})' P_2 P_1 P_2 (\Delta\hat{y})] = E(e' P_0 P_2 P_1 P_2 P_0 \varepsilon) = \text{tr}(P_0 P_2 P_1 P_2 P_0 \Sigma)
\]

\[
= \sigma_\mu^2 \text{tr}(P_0 P_2 P_1 P_2 (I_N \otimes I_T')) + \sigma_\lambda^2 \text{tr}(P_0 P_2 P_1 P_2 (i_{NI'_N} \otimes I_T')) + \sigma_v^2 \text{tr}(P_0 P_2 P_1 P_2)
\]

\[
= \sigma_\mu^2 \text{tr}(P_0 P_2 P_1 P_2 ([I_N - N^{-1}i_{NI'_N}] \otimes I_T'))
\]

\[
+ \sigma_\lambda^2 \text{tr}(P_0 P_2 P_1 P_2 [i_{NI'_N} \otimes (I_T - T^{-1}I_T I_T')]) + \sigma_v^2 \text{tr}(P_0 P_2 P_1 P_2).
\]

Since \( (I_N - N^{-1}i_{NI'_N})I_N = 0 \), we have \( P_0 P_2 P_1 P_2 = P_2 P_1 P_2 \). By using this, \( (I_N - N^{-1}i_{NI'_N})I_N = 0 \) and symmetry,

\[
E(b_0) = \sigma_\mu^2 \text{tr}(P_2 P_1 P_2 [(I_N - N^{-1}i_{NI'_N}) \otimes I_T'])
\]

\[
+ \sigma_\lambda^2 \text{tr}(P_2 P_1 P_2 [i_{NI'_N} \otimes (I_T - T^{-1}I_T I_T')]) + \sigma_v^2 \text{tr}(P_2 P_1 P_2)
\]

\[
= \sigma_\mu^2 \text{tr}(P_2 P_1 P_2) = \sigma_\mu^2 (N - 1)(T - 1).
\]

(31)

Next, consider \( E(b_1) \), which via (21) can be expanded in the following fashion:

\[
E(b_1) = E[(\Delta\hat{y})' P_2 P_1 P_2 (I_N \otimes M) (\Delta\hat{y})] = \text{tr}(P_0 P_2 P_1 P_2 (I_N \otimes M) P_0 \Sigma)
\]

\[
= \sigma_\mu^2 \text{tr}(P_2 P_1 P_2 (I_N \otimes M) [(I_N - N^{-1}i_{NI'_N}) \otimes I_T'])
\]

\[
+ \sigma_\lambda^2 \text{tr}(P_2 P_1 P_2 (I_N \otimes M) [i_{NI'_N} \otimes (I_T - T^{-1}I_T I_T')]) + \sigma_v^2 \text{tr}(P_2 P_1 P_2 (I_N \otimes M))
\]

\[
= \sigma_\mu^2 \sum_{j=1}^{T-1} \text{tr}(P_2 P_1 P_2 (I_N \otimes L^j) [(I_N - N^{-1}i_{NI'_N}) \otimes I_T'])
\]

\[
+ \sigma_\lambda^2 \sum_{j=1}^{T-1} \text{tr}(P_2 P_1 P_2 (I_N \otimes L^j) [i_{NI'_N} \otimes (I_T - T^{-1}I_T I_T')]) + \sigma_v^2 \sum_{j=1}^{T-1} \text{tr}(P_2 P_1 P_2 (I_N \otimes L^j)).
\]

Here,

\[
\text{tr}(P_2 P_1 P_2 (I_N \otimes L^j) [(I_N - N^{-1}i_{NI'_N}) \otimes I_T'])
\]

\[
= \text{tr}([I_{NT} - T^{-1}(I_N \otimes I_T I_T')][I_{NT} - N^{-1}(i_{NI'_N} \otimes I_T)]((I_N - N^{-1}i_{NI'_N}) \otimes L^j I_T I_T'))
\]

\[
= \text{tr}([I_{NT} - T^{-1}(I_N \otimes I_T I_T')][(I_N - N^{-1}i_{NI'_N}) \otimes L^j I_T I_T']) = 0,
\]

21
\[
\text{tr}(P_2P_1P_2(I_N \otimes L)[1_{N_1'} \otimes (I_T - T^{-1}t_T')]) \\
= \text{tr}([I_{NT} - T^{-1}(I_N \otimes t_T')][I_{NT} - N^{-1}(t_{NI_N} \otimes I_T)][I_{NI_N} \otimes L'(I_T - T^{-1}t_T')]) = 0,
\]
and
\[
\text{tr}[P_2P_1P_2(I_N \otimes L)] \\
= \text{tr}([I_{NT} - T^{-1}(I_N \otimes t_T')][I_{NT} - N^{-1}(t_{NI_N} \otimes I_T)](I_N \otimes L')) \\
= \text{tr}(I_N \otimes L') - N^{-1}\text{tr}(t_{NI_N} \otimes L') - T^{-1}\text{tr}(I_N \otimes t_T'L') + T^{-1}N^{-1}\text{tr}(t_{NI_N} \otimes t_T'L') \\
= -(N-1)T^{-1}(T-j),
\]
imply
\[
E(b_1) = -\sigma_v^2(N-1)T^{-1}\sum_{j=1}^{T-1}(T-j) = -\frac{1}{2}(N-1)(T-1)\sigma_v^2. \tag{32}
\]
It remains to consider \(E(b_2)\). By using calculations similar to those used in the above we can show that
\[
E(b_2) = E[(\Delta \hat{y})'(I_N \otimes M')P_2P_1P_2(I_N \otimes M)(\Delta \hat{y})] \\
= \text{tr}[(I_N \otimes M')P_2P_1P_2(I_N \otimes M)P_0\Sigma P_0] \\
= \sigma_v^2\text{tr}((I_N \otimes M')P_2P_1P_2(I_N \otimes M)[(I_N - N^{-1}t_{NI_N}) \otimes t_T']P_0) \\
+ \sigma_v^2\text{tr}((I_N \otimes M')P_2P_1P_2(I_N \otimes M)[t_{NI_N} \otimes (I_T - T^{-1}t_T')]P_0) \\
+ \sigma_v^2\text{tr}((I_N \otimes M')P_2P_1P_2(I_N \otimes M)P_0) \\
= \sigma_v^2\text{tr}((I_N \otimes M')P_2P_1P_2(I_N \otimes M)[(I_N - N^{-1}t_{NI_N}) \otimes t_T']) \\
+ \sigma_v^2\text{tr}((I_N \otimes M')P_2P_1P_2(I_N \otimes M)[t_{NI_N} \otimes (I_T - T^{-1}t_T')] \\
+ \sigma_v^2\text{tr}((I_N \otimes M')P_2P_1P_2(I_N \otimes M)P_0),
\]
where

$$\text{tr}((I_N \otimes M')P_2P_1P_2(I_N \otimes M)[(I_N - N^{-1}i_N'N) \otimes i_Ti'_T])$$

$$= \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}((I_N \otimes L^j')P_2P_1P_2[(I_N - N^{-1}i_N'N) \otimes L^kii'_T])$$

$$= \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}((I_N \otimes L^j')[I_N - T^{-1}i_Ti'_T][(I_N - N^{-1}i_N'N) \otimes L^kii'_T])$$

$$= \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}[(I_N - N^{-1}i_N'N) \otimes L^jii'_T]$$

$$- T^{-1} \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}[(I_N - N^{-1}i_N'N) \otimes L^jii'_TL^kii'_T]$$

$$= (N - 1)[\sum_{j=1}^{T-1} \sum_{k=1}^{T-1} (T - j \lor k) - T^{-1} \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} (T - j)(T - k)]$$

$$= (N - 1)[\frac{1}{6}T(T - 1)(2T - 1) - \frac{1}{4}T(T - 1)^2]$$

$$= \frac{1}{12}(N - 1)T(T^2 - 1),$$

$$\text{tr}((I_N \otimes M')P_2P_1P_2(I_N \otimes M)[i_Ni'_N \otimes (I_T - T^{-1}ii'_T)])$$

$$= \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}((I_N \otimes L^j')P_2P_1P_2[i_Ni'_N \otimes L^k(i_T - T^{-1}ii'_T)])$$

$$= \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}((I_N \otimes L^j')[I_N - T^{-1}(I_N \otimes ii'_T)][I_N - N^{-1}(i_Ni'_N \otimes I_T)]$$

$$\times [i_Ni'_N \otimes L^k(i_T - T^{-1}ii'_T)]) = 0,$$
and

\[
\text{tr}[(I_N \otimes M')P_1P_2(I_N \otimes M)P_0] \\
= \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}((I_N \otimes L')P_1P_2(I_N \otimes L^k)[I_{NT} - (NT)^{-1}t_{NT}']) \\
= \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}((I_N \otimes L')[I_{NT} - T^{-1}(I_N \otimes t_{NT}')] \\
\times [I_{NT} - N^{-1}(t_{NT} \otimes I_T)](I_N \otimes L^k)[I_{NT} - (NT)^{-1}t_{NT}'] \\
= \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} (\text{tr}((I_N \otimes L')L^k[I_{NT} - (NT)^{-1}t_{NT}'])) \\
- N^{-1}\text{tr}((t_{NTN} \otimes L'L^k)[I_{NT} - (NT)^{-1}t_{NT}']) \\
- T^{-1}\text{tr}((I_N \otimes L't_{NT}'L^k)[I_{NT} - (NT)^{-1}t_{NT}']) \\
+ (NT)^{-1}\text{tr}((t_{NTN}L't_{NT}'L^k)[I_{NT} - (NT)^{-1}t_{NT}']) \\
= \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} [N1(j = k) - T^{-1}(T - j \lor k) - 1(j = k) + T^{-1}(T - j \lor k) \\
- NT^{-1}(T - j \lor k) + T^{-2}(T - j)(T - k) + T^{-1}(T - j \lor k) - T^{-2}(T - j)(T - k) \\
= (N - 1) \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} [1(j = k) - T^{-1}(T - j \lor k) \\
= (N - 1) \left( (T - 1) - \frac{1}{6} (T - 1)(2T - 1) \right) \\
= -\frac{1}{6}(N - 1)(T - 1)(2T - 7).
\]

It follows that

\[
E(b_2) = \frac{1}{12} \sigma^2 \mu (N - 1)T(T^2 - 1) - \frac{1}{6} \sigma^2 \mu (N - 1)(T - 1)(2T - 7).
\]  \hspace{1cm} (33)

This establishes the last of the required results. The proof of the lemma is therefore complete.

\[\blacksquare\]

**Proof of Theorem 1.**

In terms of the notation introduced in Proof of Lemma 1,

\[
T\xi_1 = a_{10} - 2a_{11} \alpha + a_{12} \alpha^2, \\
N\xi_2 = a_{20} - 2a_{21} \alpha + a_{22} \alpha^2, \\
\xi_0 - \xi_1 - \xi_2 = b_0 - 2b_1 \alpha + b_2 \alpha^2.
\]
This implies
\[
\frac{d\ell_c}{d\alpha} \sim -\frac{NT}{2}(\xi_0 - \xi_1 - \xi_2)^{-1} \frac{d}{d\alpha}(\xi_0 - \xi_1 - \xi_2) - \frac{N}{2}(T\xi_1)^{-1} \frac{d}{d\alpha}(T\xi_1) - \frac{T}{2}(N\xi_2)^{-1} \frac{d}{d\alpha}(N\xi_2)
\]
\[= NT(\xi_0 - \xi_1 - \xi_2)^{-1}(b_1 - b_2a) + N(T\xi_1)^{-1}(a_{11} - a_{12}a) + T(N\xi_2)^{-1}(a_{21} - a_{22}a).
\]

We therefore need to solve
\[
0 = NT(T\xi_1)(N\xi_2)(b_1 - b_2a) + N(N\xi_2)(\xi_0 - \xi_1 - \xi_2)(a_{11} - a_{12}a) + T(T\xi_1)(\xi_0 - \xi_1 - \xi_2)(a_{21} - a_{22}a)
\]
\[= NT(a_{10} - 2a_{11}a + a_{12}a^2)(a_{20} - 2a_{21}a + a_{22}a^2)(b_1 - b_2a) + N(a_{20} - 2a_{21}a + a_{22}a^2)(a_{11} - a_{12}a)(b_0 - b_1a + b_2a^2) + T(a_{10} - 2a_{11}a + a_{12}a^2)(a_{21} - a_{22}a)(b_0 - b_1a + b_2a^2)
\]
\[\sim c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 + c_4\alpha^4 + c_5\alpha^5,
\]

where
\[
c_0 = NTa_{10}a_{20}b_1 + Na_{11}a_{20}b_0 + Ta_{10}a_{21}b_0,
\]
\[
c_1 = -NT[(a_{11}a_{20} + 2a_{10}a_{21})b_1 + a_{10}a_{20}b_2] - N[(2a_{11}a_{21} + a_{12}a_{20})b_0 + 2a_{11}a_{20}b_1]
\]
\[\quad - T[(2a_{11}a_{21} + a_{10}a_{22})b_0 + 2a_{10}a_{21}b_1],
\]
\[
c_2 = NT[(a_{12}a_{20} + 4a_{11}a_{21} + a_{10}a_{22})b_1 + (2a_{11}a_{20} + 2a_{10}a_{21})b_2] + N[(a_{11}a_{22} + 2a_{12}a_{21})b_0 + (4a_{11}a_{21} + 2a_{12}a_{20})b_1 + a_{11}a_{20}b_2]
\]
\[\quad + T[(a_{12}a_{21} + 2a_{11}a_{22})b_0 + (4a_{11}a_{21} + 2a_{10}a_{22})b_1 + a_{10}a_{21}b_2],
\]
\[
c_3 = -NT[(2a_{12}a_{21} + 2a_{11}a_{22})b_1 + (a_{12}a_{20} + 4a_{11}a_{21} + a_{10}a_{22})b_2]
\]
\[\quad - N[a_{12}a_{22}b_0 + (2a_{11}a_{22} + 4a_{12}a_{21})b_1 + (2a_{11}a_{21} + a_{12}a_{20})b_2]
\]
\[\quad - T[a_{12}a_{22}b_0 + (2a_{12}a_{21} + 4a_{11}a_{22})b_1 + (2a_{11}a_{21} + a_{10}a_{22})b_2],
\]
\[
c_4 = NT[a_{12}a_{22}b_1 + (2a_{12}a_{21} + 2a_{11}a_{22})b_2] + N[2a_{12}a_{22}b_1 + (a_{11}a_{22} + 2a_{12}a_{21})b_2]
\]
\[\quad + T[2a_{12}a_{22}b_1 + (a_{12}a_{21} + 2a_{11}a_{22})b_2],
\]
\[
c_5 = -(NT + N + T)a_{12}a_{22}b_2.
\]
According to Lemma A.1, under $H_0$, as $N \to \infty$ and/or $T \to \infty$,
\begin{align*}
c_0 & \sim Na_{20}(Ta_{10}b_1 + a_{11}b_0) = O_p(N^5T^5), \\
c_1 & \sim -N Ta_{10}a_{20}b_2 = O_p(N^5T^7), \\
c_2 & = O_p(N^5T^8), \\
c_3 & = O_p(N^5T^9), \\
c_4 & = O_p(N^5T^9 + N^4T^{10}), \\
c_5 & = O_p(N^5T^{10}).
\end{align*}

Inserting an arbitrary $\alpha$ of order $O_p(T^{-2})$, the right-hand side of (34) becomes
\begin{align*}
Na_{20}(Ta_{10}b_1 + a_{11}b_0) - NTa_{10}a_{20}b_2\alpha + O_p(N^5T^4) \\
= Na_{20}(Ta_{10}b_1 + a_{11}b_0 - Ta_{10}b_2\alpha) + O_p(N^5T^4). \tag{35}
\end{align*}

The MLE solution sets the first term to zero. Hence, when evaluated at the MLE the right-hand side of (34) is $O_p(N^5T^4)$. Let us now consider $\hat{\alpha}_{CV} = b_1/b_2 = O_p(T^{-2})$. Direct insertion into (34) yields
\begin{align*}
c_0 + c_1 \frac{b_1}{b_2} + c_2 \left(\frac{b_1}{b_2}\right)^2 + c_3 \left(\frac{b_1}{b_2}\right)^3 & = Na_{20}(Ta_{10}b_1 + a_{11}b_0) - NTa_{10}a_{20}b_1 + O_p(N^5T^4) \\
& = Na_{11}a_{20}b_0 + O_p(N^5T^4),
\end{align*}
where $Na_{11}a_{20}b_0 = O_p(N^5T^5)$. Hence, since $O_p(N^5T^5) > O_p(N^5T^4)$ for $T \to \infty$ with $N$ fixed or $N \to \infty$, $\hat{\alpha}_{CV}$ cannot be asymptotically equivalent to the MLE. However,
\begin{align*}
\hat{\alpha}_{BC} = \frac{Ta_{10}b_1 + a_{11}b_0}{Ta_{10}b_2} = \hat{\alpha}_{CV} + \frac{a_{11}b_0}{Ta_{10}b_2}
\end{align*}
makes the first in (35) disappear. $\hat{\alpha}_{BC}$ must therefore be asymptotically equivalent to the MLE (as $T \to \infty$ with $N$ fixed or $N \to \infty$), which was to be shown.

**Proof of Proposition 1.**

As we show in (41) in Proof of Theorem 2,
\begin{align*}
LR \sim N \frac{(Ta_{10}b_1 + a_{11}b_0)^2}{Ta_{10}^2b_0b_2} = N \left(\frac{Ta_{10}b_1 + a_{11}b_0}{Ta_{10}b_2}\right)^2 \frac{Tb_2}{b_0} = (\sqrt{NT}^{3/2} \hat{\alpha}_{BC})^2 \frac{b_2}{T^2b_0}. \tag{36}
\end{align*}

The proof of Proposition 1 is now an immediate consequence of Lemma A.1, Proof of Theorem 2, and the continuous mapping theorem.
Proof of Theorem 2.

Under $H_0$, $\varepsilon = P_0(\Delta y - \alpha y_-) = \varepsilon_0 = P_0\Delta y$. Also, let $\hat{\varepsilon}_1 = P_0(\Delta y - \hat{\alpha}_y y_-)$. In this notation,

$$LR = -2\ell_c(0) + 2\ell_c(\hat{\alpha}_BC).$$

$$\sim NT\log(\hat{\varepsilon}'_1 P_2 P_1 P_2 \hat{\varepsilon}_0) + \log[\hat{\varepsilon}'_0(I_N \otimes \epsilon_I T_T)\hat{\varepsilon}_0] - NT\log(\hat{\varepsilon}'_1 P_2 P_1 P_2 \hat{\varepsilon}_1)$$

$$= N(X_1 + X_2),$$

(37)

where

$$X_1 = -T\log \left( \frac{\hat{\varepsilon}'_1 P_2 P_1 P_2 \hat{\varepsilon}_0}{\hat{\varepsilon}'_0 P_2 P_1 P_2 \hat{\varepsilon}_0} \right),$$

(38)

$$X_2 = -\log \left( \frac{\hat{\varepsilon}'_0(I_N \otimes \epsilon_I T_T)\hat{\varepsilon}_0}{\hat{\varepsilon}'_0(I_N \otimes \epsilon_I T_T)\hat{\varepsilon}_0} \right)$$

(39)

Consider $X_1$. We begin by noting that if we let $d = a_{11}b_0/(Ta_{10}b_2)$, then $\varepsilon_1 = \Delta \hat{y} - \hat{\alpha}_B \hat{y}_- = \Delta \hat{y} - (\hat{\alpha}_C V_2 + d) \hat{y}_-$. By using this and the fact that $P_2 P_1$ is idempotent, we obtain

$$\hat{\varepsilon}'_1 P_2 P_1 P_2 \hat{\varepsilon}_1 = [\Delta \hat{y} - (\hat{\alpha}_C V_2 + d) \hat{y}_-[P_2 P_1 P_2[\Delta \hat{y} - (\hat{\alpha}_C V_2 + d) \hat{y}_-]$$

$$= (\Delta \hat{y})' P_2 P_1 P_2(\Delta \hat{y}) - 2\hat{\alpha}_C V_2(\Delta \hat{y})' P_2 P_1 P_2 \hat{y}_- + \hat{\alpha}_C V_2(\hat{y}_-[P_2 P_1 P_2 \hat{y}_-$$

$$- 2d[(\Delta \hat{y})' P_2 P_1 P_2 \hat{y}_- - \hat{\alpha}_C V_2(\hat{y}_-[P_2 P_1 P_2 \hat{y}_-] + d^2(\hat{y}_-[P_2 P_1 P_2 \hat{y}_-$$

$$= b_0 - \frac{b_1^2}{b_2} + \frac{a_{11}^2 b_0}{T^2 a_{10}^2 b_2}. $$

which can be inserted into (38), giving

$$X_1 = -T\log(1 - x_1),$$

where

$$x_1 = \frac{b_1^2}{b_0 b_2} - \frac{a_{11}^2 b_0}{T^2 a_{10}^2 b_2}. $$

According to Lemma A.1, $x_1 = O_p(T^{-2})$, which, via Taylor expansion, yields

$$X_1 = Tx_1 + O_p(T^{-3}).$$

(40)
As for $X_2$, we similarly have

$$
\hat{\epsilon}_1'(I_N \otimes \iota_T \iota_T')\hat{\epsilon}_1 = \left[\Delta \hat{y} - (\hat{\alpha}_{CV} + d)\hat{y}_-\right]' (I_N \otimes \iota_T \iota_T') \left[\Delta \hat{y} - (\hat{\alpha}_{CV} + d)\hat{y}_-\right] - 2 \hat{\alpha}_{CV} (\Delta \hat{y})' (I_N \otimes \iota_T \iota_T') \hat{y}_- + \hat{\alpha}_{CV}^2 (\hat{y}_-) \left(\hat{\epsilon}_1' (I_N \otimes \iota_T \iota_T') \hat{\epsilon}_1 \right) + O_p(1),
$$

where from Lemma A.1, $a_{10} = O_p(NT)$, while the second term is $O_p(NT)$. Hence,

$$
X_2 = -\log(1 - x_2 + O_p(T^{-2})),
$$

where

$$
x_2 = 2 \left( \frac{a_{11}}{a_{10}b_2} \right) \left( b_1 + \frac{a_{11}b_0}{Ta_{10}} \right),
$$

which according to Lemma A.1 is $O_p(T^{-1})$. Hence, by Taylor expansion,

$$
X_2 = x_2 + O_p(T^{-2}),
$$

which together with (40) can be inserted into (37) to obtain

$$
LR = Z_{NT} + O_p(T^{-2}),
$$

where $Z_{NT} = N(Tx_1 + x_2)$. Direct substitution of $x_1$ and $x_2$ gives

$$
N^{-1}Z_{NT} = \frac{Ta_{10}^2b_0^2}{b_0^2b_1^2} + \frac{a_{11}^2b_0}{Ta_{10}b_2} + 2 \frac{a_{11}^2b_0}{Ta_{10}b_2} \left( b_1 + \frac{a_{11}b_0}{Ta_{10}} \right) - \frac{Ta_{10}^2b_0^2}{Ta_{10}^2b_0^2} \left( \frac{Ta_{10}b_0 + a_{11}b_0}{Ta_{10}^2b_0^2} \right)^2 \left( \frac{Ta_{10}b_0 + a_{11}b_0}{Ta_{10}^2b_0^2} \right),
$$

(41)

which is positive and at most $O_p(T^{-1})$ (it will turn out to be of smaller order).

In what remains we derive the asymptotic distribution of $Z_{NT}$. The idea is to replace all quantities is (41) with their corresponding probability limits, except for $b_1$ and $a_{11}$, which are replaced by their limiting distributions. This practice is justified by the Slutsky theorem, and Corollary 1 and Theorem 3 of Phillips and Moon (1999).
We begin by considering $b_2$. From $P_0P_2P_1P_2 = P_2P_1P_2$ and the idempotency of $P_2P_1P_2$,

\[
g'_P P_2 P_1 P_2 \hat{y}_- = \epsilon' (I_N \otimes M') P_0 P_2 P_1 P_0 (I_N \otimes M) \epsilon \\
= \epsilon' (I_N \otimes M') P_2 P_1 P_2 (I_N \otimes M) \epsilon \\
= \epsilon' (I_N \otimes M') [(I_N - N^{-1}t_N'i_N') \otimes (I_T - T^{-1}t_T'i_T')] (I_N \otimes M) \epsilon \\
= \epsilon' [(I_N - N^{-1}t_N'i_N') \otimes M'(I_T - T^{-1}t_T'i_T') M] \epsilon.
\]

Note that $\epsilon = \mu \otimes t_T + t_N \otimes \lambda + \nu$, where $\mu = (\mu_1, ..., \mu_N)'$ with similar definitions of $\lambda$ and $\nu$. Thus, because $(I_N - N^{-1}t_N'i_N')1_N = 0$, the above becomes

\[
g'_P P_2 P_1 P_2 \hat{y}_- = [(\mu' \otimes I_T') + \nu'] [(I_N - N^{-1}t_N'i_N') \otimes M'(I_T - T^{-1}t_T'i_T') M] [(\mu \otimes t_T) + \nu] \\
= \mu' (I_N - N^{-1}t_N'i_N') \mu \otimes I_T' [M'(I_T - T^{-1}t_T'i_T') M] t_T \\
+ 2[\mu' (I_N - N^{-1}t_N'i_N') \otimes I_T' M'(I_T - T^{-1}t_T'i_T') M] \nu \\
+ \nu' [(I_N - N^{-1}t_N'i_N') \otimes M'(I_T - T^{-1}t_T'i_T') M] \nu. \tag{42}
\]

Here,

\[
N^{-1} T^{-3} \mu' (I_N - N^{-1}t_N'i_N') \mu \otimes I_T' [M'(I_T - T^{-1}t_T'i_T') M] t_T \\
= T^{-3} (I_T' [M'(I_T - T^{-1}t_T'i_T') M] t_T) \left[ N^{-1} \sum_{i=1}^N \mu_i^2 - \left( N^{-1} \sum_{i=1}^N \mu_i \right)^2 \right] \\
= N^{-1} \sum_{i=1}^N R_{i,T} - \left( N^{-1} \sum_{i=1}^N S_{i,T} \right)^2, \tag{43}
\]

where

\[
R_{i,T} = T^{-3} (I_T' [M'(I_T - T^{-1}t_T'i_T') M] t_T) \mu_i^2, \\
S_{i,T} = T^{-3/2} (I_T' [M'(I_T - T^{-1}t_T'i_T') M] t_T)^{1/2} \mu_i.
\]

Let us concentrate on the first term on the right-hand side of (43). The analysis of the second term is very similar. $R_{1,T}, ..., R_{N,T}$ are iid for all $T$. Corollary 1 of Phillips and Moon (1999) says that if there is a $R_i$ such that $R_{i,T} \rightarrow_d R_i$ as $T \rightarrow \infty$, and if $R_{i,T}$ is uniformly integrable in $T$ for all $i$, then

\[
N^{-1} \sum_{i=1}^N R_{i,T} \rightarrow_p E(R_i)
\]
as \( N, T \to \infty \). To establish the existence of \( R_i \), note first that \( \mu_i^2 \) does not depend on \( T \). Also,

\[
T^{-3}(i_T'[M'(I_T - T^{-1}t_T t_T')M]_{I_T}) = T^{-3}(i_T'M'M_{I_T}) - T^{-4}(i_T'M_{I_T})^2,
\]

where, by use of (21),

\[
i_T'M'M_{I_T} = \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} i_T'L_j^j L_k^k = \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} (T - j \lor k) = \frac{1}{6} T(T - 1)(2T - 1),
\]

which together with (22) yields

\[
T^{-3}(i_T'[M'(I_T - T^{-1}t_T t_T')M]_{I_T}) = \frac{1}{6} T^{-2}(T - 1)(2T - 1) - \frac{1}{2} T^{-2}(T - 1)^2
\]

\[= -\frac{1}{6} T^{-2}(T - 1)(T - 2)
\]

\[= -\frac{1}{6} + O(T^{-1}).
\]

This shows the existence of \( R_i \). Uniform integrability holds too, because for all \( T \geq 1 \),

\[
[E(|R_{i,T}|)]^2 \leq E(R_{i,T}^2) = [T^{-3}(i_T'[M'(I_T - T^{-1}t_T t_T')M]_{I_T})^2 E(\mu_i^4) \leq \frac{3\sigma_i^4}{36} = \frac{\sigma_i^4}{12}.
\]

The second and the third terms is (42) may be handled in a similar fashion. This establishes
the convergence of \( N^{-1}T^{-3}b_2 \), and it is easily seen that \( a_{10} \) and \( b_0 \) may be treated similarly.

We now justify the convergence in distribution of \( a_{11} \). By definition,

\[
a_{11} = (\Delta \hat{y})'(I_N \otimes t_T t_T')\hat{y} = \varepsilon' P_0(I_N \otimes t_T t_T') P_0(I_N \otimes M) \varepsilon,
\]

where

\[
P_0(I_N \otimes t_T t_T') P_0 = [(I_N - N^{-1}t_N t_N' N I_{NT}) \otimes t_T t_T'][I_{NT} - (N T)^{-1} \otimes t_T t_T']
\]

\[= (I_N - N^{-1}t_N t_N') \otimes t_T t_T' = \bar{P}.
\]

Hence,

\[
a_{11} = \varepsilon' \bar{P}(I_N \otimes M) \varepsilon
\]

\[= [(\mu' \otimes \mu_t') + \nu']\bar{P}(I_N \otimes M)[(\mu \otimes \mu_t) + \nu]
\]

\[= (\mu' \otimes \mu_t')\bar{P}(I_N \otimes M)(\mu \otimes \mu_t) + 2(\mu' \otimes \mu_t')\bar{P}(I_N \otimes M)\nu + \nu' \bar{P}(I_N \otimes M) \nu.
\]

For the first term on the right, letting \( R_{i,T}^* = T^{-2}(i_T'M_{I_T}) \mu_i^2 \) and \( S_{i,T}^* = \sqrt{T^{-2}(i_T'M_{I_T})} \mu_i \),
we have

\[
T^{-3}(\mu' \otimes \mu_t')\bar{P}(I_N \otimes M)(\mu \otimes \mu_t) = T^{-2}[\mu'(I_N - N^{-1}t_N t_N') \mu](i_T'M_{I_T})
\]

\[= \sum_{i=1}^{N} R_{i,T}^* - N^{-1} \left( \sum_{i=1}^{N} S_{i,T}^* \right)^2.
\]
As in the case of $R_{i,T}$, we can show that $N^{-1} \sum_{i=1}^{N} R_{i,T}^{*}$ converges in probability. In fact,

$$N^{-1} \sum_{i=1}^{N} R_{i,T}^{*} \to_{p} \frac{(T-1)^2}{2T^2} \sigma_{\mu}^2$$

as $N, T \to \infty$. To also show convergence in distribution, we need to verify that the variance of $R_{i,T}^{*}$ is finite for all $T$ (Phillips and Moon, 1999, Theorem 3). But

$$\sigma_{R}^2 = \text{var}(R_{i,T}^{*}) = \frac{(T-1)^4}{4T^4} \sigma_{\mu}^4 < \frac{3\sigma_{\mu}^4}{4}$$

for all $T$, and so

$$N^{-1/2} \sum_{i=1}^{N} \left( R_{i,T}^{*} - \frac{(T-1)^2}{2T^2} \sigma_{\mu}^2 \right) \to_{d} N(0, \sigma_{R}^2)$$

as $N, T \to \infty$. $N^{-1/2} \sum_{i=1}^{N} S_{i,T}^{*}$ is asymptotically normal too. Hence,

$$N^{-1/2} \left( T^{-3} (\mu' \otimes \iota_T') P(I_N \otimes M)(\mu \otimes \iota_T) - N^2 \frac{(T-1)^2}{2T^2} \sigma_{\mu}^2 \right)$$

$$= N^{-1/2} \sum_{i=1}^{N} \left( R_{i,T}^{*} - \frac{(T-1)^2}{2T^2} \sigma_{\mu}^2 \right) - N^{-1/2} \left( N^{-1/2} \sum_{i=1}^{N} S_{i,T}^{*} \right)^2 \to_{d} N(0, \sigma_{R}^2)$$

as $N, T \to \infty$. The second and third terms in (46) may be treated similarly, and a similar argument goes through also for $b_1$. This verifies that all quantities is (41) may be replaced by their probability limits, except for $b_1$ and $a_{11}$, which may be replaced by their limiting distributions.

To find the distribution of $Z_{NT}$, note that by Lemma A.1,

$$N^{-1} T^{-2} a_{10} = \sigma_{\mu}^2 + O_p(T^{-1}),$$

$$(NT)^{-1} b_0 = \sigma_{\nu}^2 + O_p(N^{-1}) + O_p(T^{-1}),$$

$$N^{-1} T^{-3} b_2 = \frac{\sigma_{\nu}^2}{12} + O_p(T^{-3}) + O_p(N^{-1} T^{-2}),$$

so that (41) yields

$$Z_{NT} = N \frac{(Ta_{10} b_1 + a_{11} b_0)^2}{Ta_{10}^2 b_0 b_2}$$

$$= 12N \frac{(NT^3 \sigma_{\mu}^2 b_1 + a_{11} NT \sigma_{\nu}^2)^2}{N^4 T^9 \sigma_{\mu}^6 \sigma_{\nu}^2} + O_p(N^{-1}) + O_p(T^{-1})$$

$$= N^{-1} T^{-3} \frac{12}{\sigma_{\mu}^6 \sigma_{\nu}^2} U^2 + O_p(N^{-1}) + O_p(T^{-1}),$$

(47)
where, letting \( B = \sigma_\mu^2 P_2 P_1 P_2 + T^{-2} \sigma_v^2 (I_N \otimes \iota_T \iota_T') \),

\[
U = \sigma_\mu^2 b_1 + T^{-2} \sigma_v^2 a_{11} = (\Delta \hat{y})' B \hat{y} - (\Delta y)' P_0 BP_0 y - .
\] (48)

Let \( \overline{B} = P_0 BP_0 \). From (16), (45) and \( P_0 P_2 P_1 P_2 = P_2 P_1 P_2 \),

\[
\overline{B} = \sigma_\mu^2 P_2 P_1 P_2 + T^{-2} \sigma_v^2 \overline{P}.
\] (49)

In view of (48), because \( b_1 \) and \( a_{11} \) are asymptotically normal, we only need to find the mean and variance of \( U \). As for the mean, by Lemma A.1,

\[
E(U) = \sigma_\mu^2 E(b_1) + T^{-2} \sigma_v^2 E(a_{11})
\]

\[
= -\frac{1}{2} (N - 1)(T - 1) \sigma_\mu^2 \sigma_v^2 + \frac{1}{2} (N - 1) T^{-1} (T - 1) \sigma_v^2 (\sigma_v^2 + T \sigma_\mu^2)
\]

\[
= \frac{1}{2} (N - 1) T^{-1} (T - 1) \sigma_v^4
\]

\[
= \frac{(N - 1) \sigma_v^4}{2} + O_p(NT^{-1}).
\] (50)

To find the variance, we use (48) and (49) to write

\[
U = \epsilon' \overline{B}(I_N \otimes M) \epsilon = \epsilon' \overline{C} \epsilon,
\] (51)

with and obvious definition of \( \overline{C} \). We now want to make use of the result that \( \text{var}(\epsilon' A \epsilon) = 2\text{tr}[(A \Sigma)^2] \) for a symmetric matrix \( A \). This requires us to symmetrize \( \overline{C} \). That is, we need to find a symmetric matrix \( \overline{C} \) such that \( \epsilon' C \epsilon = \epsilon' \overline{C} \epsilon \). But since \( \overline{B} \) is symmetric, we may set

\[
\overline{C} = \frac{1}{2} \overline{B}(I_N \otimes M) + (I_N \otimes M') \overline{B}.
\]

Observe that \( \overline{C}(I_N \iota_N' \otimes I_T) = 0 \), because

\[
\overline{B}(I_N \otimes M)(I_N \iota_N' \otimes I_T) = (\sigma_\mu^2 P_2 P_1 P_2 + T^{-2} \sigma_v^2 \overline{P})(I_N \iota_N' \otimes M),
\]

where

\[
P_2 P_1 P_2 (I_N \iota_N' \otimes M) = [(I_N - N^{-1} I_N \iota_N') I_N \iota_N'] \otimes [(I_T - T^{-1} I_T \iota_T') M] = 0,
\]

\[
\overline{P}(I_N \iota_N' \otimes M) = [(I_N - N^{-1} I_N \iota_N') I_N \iota_N] \otimes (I_T \iota_T' M) = 0.
\]

Similarly,

\[
(I_N \otimes M') \overline{B}(I_N \iota_N' \otimes I_T) = (I_N \otimes M')(\sigma_\mu^2 P_2 P_1 P_2 + T^{-2} \sigma_v^2 \overline{P})(I_N \iota_N' \otimes I_T) = 0.
\]
These results, together with the definition of $\Sigma$, imply

$$
\text{tr}[(\overline{C}\Sigma)^2] = \text{tr}[(\sigma^2_T \overline{C}(I_N \otimes i_T I_T') + \sigma^2_T \overline{C}(I_N i_N' \otimes I_T) + \sigma^2_T \overline{C})^2]
$$

$$= \text{tr}[(\sigma^2_T \overline{C}(I_N \otimes i_T I_T') + \sigma^2_T \overline{C})^2]
$$

$$= \sigma^4_T \text{tr}[(\overline{C}(I_N \otimes i_T I_T'))^2] + 2\sigma^2_T \sigma^2_T \text{tr}[\overline{C}^2(I_N \otimes i_T I_T')] + \sigma^4_T \text{tr}[(\overline{C})^2],
$$

(52)

where, because $\text{tr}[(AB)^2] = \text{tr}[(AB')^2]$, where $A$ is symmetric,

$$
\text{tr}[(\overline{C}(I_N \otimes i_T I_T'))^2] = \frac{1}{4} \text{tr}[(\overline{B}(I_N \otimes M) + (I_N \otimes M')(\text{tr} I_T'))^2]
$$

$$= \frac{1}{2} \text{tr}[(\overline{B}(I_N \otimes M)(I_N \otimes i_T I_T'))^2]
$$

$$+ \frac{1}{2} \text{tr}[(\overline{B}(I_N \otimes M)(I_N \otimes i_T I_T')(I_N \otimes M')\overline{B}(I_N \otimes i_T I_T')].
$$

(53)

Since $(I_N \otimes i_T I_T')P_2 P_1 P_2 = (I_N - N^{-1}i_N') \otimes [i_T I_T'(I_T - T^{-1}i_T')] = 0$, we have, using (22),

$$
\text{tr}[(\overline{B}(I_N \otimes M)(I_N \otimes i_T I_T'))^2] = \text{tr}[(\overline{B}(I_N \otimes M)(I_N \otimes i_T I_T')^2]
$$

$$= T^{-4} \sigma^4_T \text{tr}[(\overline{B}(I_N \otimes M)(I_N \otimes i_T I_T'))^2]
$$

$$= T^{-4} \sigma^4_T \text{tr}[(I_N - N^{-1}i_N') \otimes i_T I_T'M(i_T I_T')]^2]
$$

$$= T^{-2} \sigma^4_T (N - 1)(i_T M)^2
$$

$$= \frac{1}{4}(N - 1)(T - 1)^2 \sigma^4_T,
$$

and

$$
\text{tr}[(\overline{B}(I_N \otimes M)(I_N \otimes i_T I_T')(I_N \otimes M')(\overline{B}(I_N \otimes i_T I_T']
$$

$$= \text{tr}[(\sigma^2_T P_2 P_1 P_2 + T^{-2} \sigma^2_T \overline{P})(I_N \otimes M i_T I_T')M'(\sigma^2_T P_2 P_1 P_2 + T^{-2} \sigma^2_T \overline{P})(I_N \otimes i_T I_T')]
$$

$$= T^{-4} \sigma^4_T \text{tr}[(\overline{P}(I_N \otimes M i_T I_T')M'\overline{P}(I_N \otimes i_T I_T')]
$$

$$= T^{-4} \sigma^4_T \text{tr}[(I_N - N^{-1}i_N') \otimes i_T I_T'M(i_T I_T')]
$$

$$= T^{-2} \sigma^4_T (N - 1)(i_T M)^2
$$

$$= \frac{1}{4}(N - 1)(T - 1)^2 \sigma^4_T,
$$

which, when inserted into (53), yield

$$
\text{tr}[(\overline{C}(I_N \otimes i_T I_T'))^2] = \frac{1}{4}(N - 1)(T - 1)^2 \sigma^4_T.
$$

(54)
Moreover,
\[
\text{tr}[C^2(I_N \otimes IT')]
\]
\[
= \frac{1}{4} \text{tr}[(B(I_N \otimes M) + (I_N \otimes M')B)^2(I_N \otimes IT')]
\]
\[
= \frac{1}{2} \text{tr}[(B(I_N \otimes M))^2(I_N \otimes IT')] + \frac{1}{4} \text{tr}[B(I_N \otimes M)(I_N \otimes M')B(I_N \otimes IT')]
\]
\[
+ \frac{1}{4} \text{tr}[(I_N \otimes M')B^2(I_N \otimes M)(I_N \otimes IT')],
\]
where
\[
\text{tr}[(B(I_N \otimes M))^2(I_N \otimes IT')]
\]
\[
= \text{tr}[(\sigma^2 P_1 P_2 + T^{-2}\sigma^2 \bar{P})(I_N \otimes M)^2(I_N \otimes IT')]
\]
\[
= T^{-2}\sigma^2 \text{tr}[\bar{P}(I_N \otimes M)(\sigma^2 P_1 P_2 + T^{-2}\sigma^2 \bar{P})(I_N \otimes M')IT']
\]
\[
= T^{-2}\sigma^2 \text{tr}[(I_N - N^{-1}l_T') \otimes IT'TM(I_T - T^{-1}l_T')M'T]
\]
\[
+ T^{-2}\sigma^2 \text{tr}[(I_N - N^{-1}l_T') \otimes IT'T'M'Tl_T]
\]
\[
= (N - 1)T^{-2}\sigma^2[T\sigma^2((l_T'M') - T^{-1}(l_T'M)^2) + T^{-1}\sigma^2(l_T'M)^2].
\]
From (21), because \(l_T' L/k l_T = 0\) if \(T - j < k\) and \(T - j - k\) otherwise,
\[
l_T'M^2l_T = \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} l_T' L/k l_T = \sum_{j=1}^{T-1} \sum_{k=1}^{T-j} (T - j - k) = \frac{1}{6}T(T - 1)(T - 2),
\]
and so, via (22),
\[
\text{tr}[B(I_N \otimes M)]^2(I_N \otimes IT')
\]
\[
= (N - 1)T^{-2}\sigma^2 \left[ T\sigma^2 \left( \frac{1}{6}T(T - 1)(T - 2) - \frac{1}{4}T(T - 1)^2 \right) + \sigma^2 \frac{1}{4}T(T - 1)^2 \right]
\]
\[
= (N - 1)T^{-1}\sigma^2 \left( - \frac{1}{12}T(T^2 - 1)\sigma^2 + \frac{1}{4}(T - 1)^2\sigma^2 \right).
\]
We similarly find
\[
\text{tr}[B(I_N \otimes M)(I_N \otimes M')B(I_N \otimes IT')]
\]
\[
= \text{tr}[(\sigma^2 P_1 P_2 P_1 P_2 + T^{-2}\sigma^2 \bar{P})(I_N \otimes MM')(\sigma^2 P_1 P_2 P_1 P_2 + T^{-2}\sigma^2 \bar{P})(I_N \otimes IT')]
\]
\[
= T^{-4}\sigma^4 \text{tr}[\bar{P}(I_N \otimes MM')(\sigma^2 P_1 P_2 P_1 P_2 + T^{-2}\sigma^2 \bar{P})(I_N \otimes IT')]
\]
\[
= T^{-4}\sigma^4 \text{tr}[(I_N - N^{-1}l_T') \otimes IT'T'M'M'I'T']
\]
\[
= (N - 1)T^{-2}\sigma^4(l_T'M'T),
\]
where, using (21) and (26),

\[
\iota_T'MM'\iota_T = T - \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} i_j I^k I = \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} (T - j \vee k) = \frac{1}{6}T(T-1)(2T-1), \tag{57}
\]

implying

\[
\text{tr}[\mathcal{B}(I_N \otimes M)(I_N \otimes \iota_T'\iota_T'\iota_T)\mathcal{B}(I_N \otimes \iota_T'\iota_T)] = \frac{1}{6}(N - 1)T^{-1}(T-1)(2T-1)\sigma_v^4.
\]

Furthermore,

\[
\text{tr}[(I_N \otimes M')\mathcal{B}^2(I_N \otimes M)(I_N \otimes \iota_T'\iota_T)]
\]

\[
= \text{tr}[(\sigma_p^2 P_2 P_1 + T^{-2} \sigma_v^2 \mathcal{B})^2(I_N \otimes M_t'\iota_T'M')] \\
= \sigma_p^4 \text{tr}[(I_N - N^{-1}t_N'\iota_N') \otimes (I_T - T^{-1}t_T'i_T') M_t'\iota_T'M'] \\
+ 2T^{-2} \sigma_v^4 \sigma_p^2 \text{tr}[(I_N - N^{-1}t_N'\iota_N') \otimes (I_T - T^{-1}t_T'i_T') t_T'i_T'M_t'\iota_T'M'] \\
+ T^{-4} \sigma_v^4 \text{tr}[(I_N - N^{-1}t_N'\iota_N') \otimes (t_T'i_T')^2 M_t'\iota_T'M'] \\
= (N - 1)[\sigma_p^4((\iota_T'MM'\iota_T) - T^{-1}(\iota_T'M_t)^2) + T^{-3} \sigma_v^4(\iota_T'M_t)^2],
\]

where, via (57),

\[
\text{tr}[(I_N \otimes M')\mathcal{B}^2(I_N \otimes M)(I_N \otimes \iota_T'\iota_T)] \\
= (N - 1) \left[ \frac{1}{6} T(T-1)(2T-1) - \frac{1}{4} (T-1)^2 \right] + \sigma_p^4 \frac{1}{4} T^{-1}(T-1)^2 \\
= (N - 1) \left( \frac{1}{12} T(T^2 - 1)\sigma_p^4 + \frac{1}{4} T^{-1}(T-1)^2\sigma_v^4 \right).
\]

Insertion into (55) now yields

\[
\text{tr}[(\sigma_pm^2(I_N \otimes \iota_T'\iota_T)] \\
= \frac{1}{2} (N - 1)T^{-1}\sigma_p^2 \left( -\frac{1}{12} T(T^2 - 1)\sigma_p^2 + \frac{1}{4} (T-1)^2\sigma_v^2 \right) \\
+ \frac{1}{24} (N - 1)T^{-1}(T-1)(2T-1)\sigma_p^4 \\
+ \frac{1}{4} (N - 1) \left( \frac{1}{12} T(T^2 - 1)\sigma_p^4 + \frac{1}{4} T^{-1}(T-1)^2\sigma_v^4 \right) \\
= \frac{1}{48} (N - 1)[T(T^2 - 1)\sigma_p^4 - 2(T^2 - 1)\sigma_p^2\sigma_v^2 + T^{-1}(13T - 11)(T-1)\sigma_v^4]. \tag{58}
\]

Finally,

\[
\text{tr}(\sigma_p^2) = \frac{1}{4} \text{tr}[(\mathcal{B}(I_N \otimes M) + (I_N \otimes M')\mathcal{B})^2] \\
= \frac{1}{2} \text{tr}[(\mathcal{B}(I_N \otimes M))^2] + \frac{1}{2} \text{tr}[\mathcal{B}^2(I_N \otimes MM')], \tag{59}
\]

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where
\[
\text{tr}[(B(N \otimes M))^2] = \text{tr}[((\sigma^2_\mu P_1 P_2 + T^{-2} \sigma^2_\tau P) (N \otimes M))^2]
\]
\[
= \sigma^4_\mu \text{tr}[(P_1 P_2 (N \otimes M))^2] + 2T^{-2} \sigma^2_\mu \sigma^2_\tau \text{tr}[P_1 P_2 (N \otimes M)\overline{P}(N \otimes M)]
\]
\[
+ T^{-4} \sigma^4_\mu \text{tr}[(\overline{P}(N \otimes M))^2].
\]  
(60)

Here,
\[
\text{tr}[(P_1 P_2 (N \otimes M))^2] = \text{tr}[((N - N^{-1} i_N l_N') \otimes (I_T - T^{-1} i_T l_T') M)^2]
\]
\[
= (N - 1) \text{tr}[((I_T - T^{-1} i_T l_T') M)^2]
\]
\[
= (N - 1) [\text{tr}(M^2) - 2T^{-1}(l_T' M^2 i_T) + T^{-2}(l_T' M i_T)^2],
\]

where \(\text{tr}(M^2) = \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}(U/L)^k = 0\), which together with (22) and (56) yields
\[
\text{tr}[(P_1 P_2 (N \otimes M))^2] = (N - 1) \left(-\frac{1}{3}(T - 1)(T - 2) + \frac{1}{4}(T - 1)^2\right)
\]
\[
= -\frac{1}{12}(N - 1)(T - 1)(T - 5).
\]

Also, from (22) and (56),
\[
\text{tr}[P_1 P_2 (N \otimes M)\overline{P}(N \otimes M)] = \text{tr}[((N - N^{-1} i_N l_N') \otimes (I_T - T^{-1} i_T l_T') M i_T l_T M)]
\]
\[
= (N - 1) i_T' M (I_T - T^{-1} i_T l_T') M i_T
\]
\[
= (N - 1) \left(\frac{1}{6}T(T - 1)(T - 2) - \frac{1}{4}T(T - 1)^2\right)
\]
\[
= -\frac{1}{12}(N - 1)T(T^2 - 1),
\]

and
\[
\text{tr}[(\overline{P}(N \otimes M))^2] = \text{tr}[((N - N^{-1} i_N l_N') \otimes i_T l_T M)^2]
\]
\[
= (N - 1)(i_T' M i_T)^2 = \frac{1}{4}(N - 1)T^2(T - 1)^2.
\]

By inserting these results into (60), we obtain
\[
\text{tr}[(B(N \otimes M))^2] = -\frac{1}{12}(N - 1)(T - 1)(T - 5)\sigma^4_\mu - \frac{1}{6}(N - 1)T^{-1}(T^2 - 1)\sigma^2_\mu \sigma^2_\tau
\]
\[
+ \frac{1}{4}(N - 1)T^{-2}(T - 1)^2 \sigma^4_\tau.
\]  
(61)
Furthermore,
\[
\begin{align*}
\text{tr}[\overline{B}^2(I_N \otimes MM')] &= \text{tr}[(\sigma^2_{\mu} P_2 P_1 P_2 + T^{-2} \sigma^2_{\nu} \overline{P})^2(I_N \otimes MM')] \\
&= \sigma^4_{\mu} \text{tr}[P_2 P_1 P_2 (I_N \otimes MM')] + 2T^{-2} \sigma^2_{\mu} \sigma^2_{\nu} \text{tr}[P_2 P_1 P_2 \overline{P} (I_N \otimes MM')] \\
&+ T^{-4} \sigma^4_{\nu} \text{tr}[^{\overline{P}}^2 (I_N \otimes MM')],
\end{align*}
\]

where
\[
\text{tr}[P_2 P_1 P_2 (I_N \otimes MM')] = \text{tr}[(I_N - N^{-1} i_{NI_N}) \otimes (I_T - T^{-1} i_{IT_T}) MM']
\]
\[
= (N - 1) \text{tr}[(I_T - T^{-1} i_{IT_T}) MM']
\]
\[
= (N - 1) [\text{tr}(MM') - T^{-1} (i_{IT_T} MM')].
\]

Here,
\[
\text{tr}(MM') = \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \text{tr}(L/L^k) = \sum_{j=1}^{T-1} \text{tr}(L'/L') = \sum_{j=1}^{T-1} (T - j) = \frac{1}{2} T(T - 1),
\]
and so, via (57),
\[
\text{tr}[P_2 P_1 P_2 (I_N \otimes MM')] = (N - 1) \left( \frac{1}{2} T(T - 1) - \frac{1}{6} (2T - 1)(T - 1) \right) = \frac{1}{6} (N - 1)(T^2 - 1).
\]
Furthermore,
\[
\text{tr}[P_2 P_1 P_2 \overline{P} (I_N \otimes MM')] = \text{tr}[(I_N - N^{-1} i_{NI_N}) \otimes (I_T - T^{-1} i_{IT_T}) (i_{IT_T} MM')] = 0,
\]
and using again (57),
\[
\text{tr}[^{\overline{P}}^2 (I_N \otimes MM')] = \text{tr}[(I_N - N^{-1} i_{NI_N}) \otimes (i_{IT_T} T^2 MM')] = 0.
\]

By inserting the above results into (62), we get
\[
\text{tr}[\overline{B}^2(I_N \otimes MM')] = \frac{1}{6} (N - 1)[(T^2 - 1) \sigma^4_{\mu} + T^{-2} (2T - 1)(T - 1) \sigma^4_{\nu}],
\]
which in turn implies, via (59) and (61),
\[
\text{tr}(C^2) = -\frac{1}{24} (N - 1)(T - 1)(T - 5) \sigma^4_{\mu} - \frac{1}{12} (N - 1) T^{-1} (T^2 - 1) \sigma^2_{\nu} \sigma^2_{\nu} \\
+ \frac{1}{8} (N - 1) T^{-2} (T - 1)^2 \sigma^4_{\nu} + \frac{1}{12} (N - 1) [(T^2 - 1) \sigma^4_{\mu} + T^{-2} (2T - 1)(T - 1) \sigma^4_{\nu}] \\
= \frac{1}{24} (N - 1)(T + 7)(T - 1) \sigma^4_{\mu} - \frac{1}{12} (N - 1) T^{-1} (T^2 - 1) \sigma^2_{\nu} \sigma^2_{\nu} \\
+ \frac{1}{24} (N - 1)(T - 2)(T - 7) \sigma^4_{\nu}. 
\]
In order to obtain \( \text{var}(U) \), we first substitute (54) and (58) into (52). We then use the resulting expression, (51) and \( \text{var}(\epsilon'\epsilon) = 2\text{tr}[(A\Sigma)^2] \) to obtain
\[
\text{var}(U) = \frac{1}{2} (N - 1)(T - 1)^2 \sigma^4 + \frac{1}{12} (N - 1) \sigma^2 \sigma^2 (T(T - 1) \sigma^4 - 2(T^2 - 1) \sigma^2 + T^{-1}(13T - 11)(T - 1) \sigma^4) + \frac{1}{12} (N - 1)(T + 7)(T - 1) \sigma^4 \sigma^4 - \frac{1}{6} (N - 1) T^{-1}(T^2 - 1) \sigma^2 \sigma^6 + \frac{1}{12} (N - 1) T^{-2}(7T - 5)(T - 1) \sigma^8
\]
\[= \frac{1}{12} (N - 1)(T - 1)[T(T + 1) \sigma^4 + (5T - 1) \sigma^4 \sigma^4 + T^{-1}(11T - 13) \sigma^2 \sigma^6 + T^{-2}(7T - 5) \sigma^8]
\]
\[= \frac{1}{12} NT^3 \sigma^6 \sigma^2 + O(NT^2).
\]
By using this, and the asymptotic normality of \( b_1 \) and \( a_{11} \),
\[N^{-1/2}T^{-3/2}[U - E(U)] \rightarrow_d \frac{1}{\sqrt{12}} \sigma^2 \sigma_v \nu
\]as \( N, T \rightarrow \infty \), where \( \nu \sim N(0,1) \). Thus, in conjunction with (50),
\[N^{-1/2}T^{-3/2}U = N^{-1/2}T^{-3/2}E(U) + N^{-1/2}T^{-3/2}[U - E(U)]
\]
\[= N^{-1/2}T^{-5/2}(N - 1)(T - 1) \sigma^4 + N^{-1/2}T^{-3/2}[U - E(U)]
\]
and so
\[N^{-1/2}T^{-3/2}U \sim N^{-1/2}T^{-5/2}(N - 1)(T - 1) \sigma^4 + \frac{1}{\sqrt{12}} \sigma^2 \sigma_v \nu,
\]which via (47) in turn implies
\[Z_{NT} = \frac{12}{\sigma^2 \sigma_v} (N^{-1/2}T^{-3/2}U)^2 + O_p(N^{-1}) + O_p(T^{-1})
\]
\[\sim \frac{12}{\sigma^2 \sigma_v^2} \left( N^{-1/2}T^{-5/2}(N - 1)(T - 1) \sigma^4 + \frac{1}{\sqrt{12}} \sigma^2 \sigma_v \nu \right)^2 + O_p(N^{-1}) + O_p(T^{-1})
\]
\[= \left( \sqrt{NT^{-3/2}} \sqrt{3r \sigma^2} + \nu \right)^2 + O_p(\sqrt{NT^{-5/2}}) + O_p(N^{-1}) + O_p(T^{-1}). \quad (63)
\]
where \( r = \sigma^2 / \sigma^2 \). If \( N/T^3 \rightarrow 0 \), then
\[Z_{NT} \sim [\nu + o(1)]^2 + o_p(1) \rightarrow_d \nu^2 \sim \chi^2(1).
\]
On the other hand, if \( N/T^3 \rightarrow c > 0 \), we have from (63) that
\[Z_{NT} \sim \left( \sqrt{3cr^{-3/2}} + \nu \right)^2 + o_p(1),
\]
or
\[
(\sqrt{Z_{NT}} - \sqrt{3r_{\mu}^{-3/2}})^2 \to_d v^2 \sim \chi^2(1).
\]

Finally, if \( N/T^3 \to \infty \) but \( N/T^5 \to 0 \), then
\[
Z_{NT} \sim NT^{-3} [3r_{\mu}^{-3/2} + o(1)]^2 + o_p(1) \sim NT^{-3} 3r_{\mu}^{-3}
\]
and
\[
Z_{NT} - 3NT^{-3}r_{\mu}^{-3} \sim \sqrt{12NT^{-3/2}r_{\mu}^{-3/2}} v,
\]
implying
\[
\frac{(Tr_{\mu})^{3/2}}{\sqrt{12N}} (Z_{NT} - 3NT^{-3}r_{\mu}^{-3}) \to_d v,
\]
or
\[
\frac{(Tr_{\mu})^3}{12N} (Z_{NT} - 3NT^{-3}r_{\mu}^{-3})^2 \to_d v^2 \sim \chi^2(1).
\]

This completes the proof.
Table 1: Bias and RMSE of $\hat{\alpha}_{CV}$ and $\hat{\alpha}_{BC}$ when $\alpha = 0$.

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Table 2: 5% size and power.

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Notes: $LR_1$, $LR_2$ and $LR_3$ refer to the $LR$ statistic based on Theorem 2 (i)–(iii), respectively. A zero superscript signifies that $r_\mu$ is treated as known. No superscript means that the test statistic is based on using $\hat{r}_\mu$ in place of $r_\mu$. 