Sequential Auctions, Price Trends, and Risk Preferences

Audrey Hu

*University of Amsterdam/Tinbergen Institute*
Email: x.hu@uva.nl

Liang Zou

*University of Amsterdam*
Email: l.zou@uva.nl

November 6, 2014
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By Audrey Hu and Liang Zou

Abstract

We analyze sequential auctions where bidders are heterogeneous in risk exposures and exhibit non-quasilinear utilities. We derive an increasing pure strategy equilibrium for the sequential Dutch and Vickrey auctions respectively, with an arbitrary number of identical objects for sale. A sufficient, and to certain extent necessary, condition for this result is that bidders’ marginal utilities are log-submodular in income and type. This condition is fairly general, and in the environment we consider implies that both the Dutch and Vickrey sequential auctions are ex post efficient. We then show that when bidders are risk averse (preferring), the equilibrium price sequences must be downward (upward) drifting. In particular, the “declining price anomaly” is perfectly consistent with nonincreasing absolute risk aversion when bidders have exposures to background risks—that is, when failure of acquiring the auctioned object entails negative consequences.

Key words: sequential auction, background risk, risk preference, declining price, log-submodularity, ex post efficiency

JEL classification: D44, D82
1 Introduction

Sequential auctions frequently take place to sell multiple units of similar objects—one after another—using the same auction policy. Examples range from fine wine, cut flowers, live cattle, licenses, mineral rights to blocks of shares of IPO firms or the like. Bidders at these auctions are typically businesspersons to whom both winning and losing can have risky consequences. For example, it can be a firm bidding for an asset to diversify its ongoing risk, or a wholesaler for roses to supply foreign demand and so on. In these situations, a bidder’s willingness-to-pay can be directly related to the severity of the undesirable consequences should he lose, as well as the added values should he win. We construe these situations as bidders having exposures to the background risk.\textsuperscript{1}

The literature on sequential auctions has been so far focused on riskless pay-offs or risk neutral bidders. Important issues as to how bidders’ risk preferences would affect competitive bidding strategies, behavior of price patterns, and ex post efficiency, in general, remain highly conjectural.

To take a serious look at these problems, the present study considers a general model of sequential auctions in which bidders can have privately known exposures to background risk, and can be risk averse, risk neutral, or risk preferring. We focus on a sequence of Dutch or Vickrey auctions with an arbitrary number $m$ of identical objects for sale to $n$ ($> m$) competing bidders, each having a unit demand (e.g., Milgrom and Weber, 2000; Krishna, 2010). Unlike single-unit auctions, in sequential auctions bidders have the option to buy the object in any period of the auctions and therefore a new dimension of strategic decision arises: now or later? The rational trade-off calls on a bidder to weigh his utility of winning in the current

\textsuperscript{1}Some studies allow bidders to have exposures to ensuing risk, or \textit{ex post risk}, upon winning—i.e., the true value of the object, or its contribution to the bidder’s payoff, remains uncertain when the auction concludes (e.g., Maskin and Riley, 1984; Eso and White, 2004; Hu, Matthews and Zou, 2014; and Hu, Offerman and Zou, 2014). Our notion of background risk incorporates ensuing risk as a special case, but in general also allows losing bidders to face undesirable risky consequences.
period against his expected utility of winning in the subsequent periods—given the
bid history, the bidder’s future plan, and the equilibrium play of the others. Risk
aversion complicates such a trade-off and, as shown in McAfee and Vincent (1993),
in the standard symmetric private values model the existence of a pure strategy
equilibrium rests on the assumption that bidders exhibit nondecreasing absolute
risk aversion (NDARA). When bidders exhibit nonincreasing absolute risk aversion
(NIARA), an equilibrium may have to involve mixed strategies that are ex post
inefficient. The case beyond two units for sale is left open because of the difficulty
in establishing equilibria for \( m \)-period auctions even under the NDARA preferences.

A contribution of the present study is to show that the existence of a pure
strategy equilibrium hinges on a key assumption that bidders’ marginal utilities are
log-submodular in income and type or, equivalently, log-supermodular in payment
and type (see, e.g., Jewitt, 1987; Milgrom and Roberts, 1990a, 1990b; Vives, 1999,
2005; Athey, 2001, 2002). This condition is fairly general, as it imposes no restric-
tion on the sign of the Arrow-Pratt measure of absolute risk aversion. Yet, the
condition provides a powerful tool that allows us to simplify and clarify the analysis
of sequential auctions, with more general results. We summarize a number of useful
properties of log-submodular marginal utility functions in Appendix A, and derive
a unique pure strategy symmetric equilibrium under this condition in the sequential
Dutch and Vickrey auctions, respectively, for the general \( m \)-period case (Proposi-
tions 1 and 2). As a result, both the Dutch and the Vickrey sequential auctions are
ex post efficient in the general environment considered in this paper.

Another contribution of this study relates to the well-known “declining price
anomaly,” which has received much academic attention after Ashenfelter’s (1989)
documentation of the “afternoon effect” in wine auctions. The “anomaly” refers to

\[^2\] A good number of papers have since emerged seeking particular assumptions or institutional
details that may rationalize the declining price phenomenon. E.g., Black and De Meza (1992);
McAfee and Vincent (1993, 1997); Bernhardt and Scoones (1994); Engelbrecht-Wiggans (1994);
Gale and Hausch (1994); Menezes and Monteiro (2003); Von der Fehr (1994); Jeitschko (1999);
the empirical observations of downward-drifting price patterns for similar objects sold in sequential auctions.\textsuperscript{3} It contradicts the standard theoretical predictions in the risk neutral paradigm that the expected prices should be the same when bidders have private values, or increasing when bidders’ valuations are affiliated (e.g., Milgrom and Weber, 2000; Weber, 1983). Intuitively, risk aversion offers a simple explanation as to why the equilibrium prices may exhibit downward drifting patterns (e.g., Ashenfelter, 1989; McAfee and Vincent, 1993; Milgrom and Weber, 2000; Mezzetti, 2011). This is because the risk averse bidders are willing to pay a premium for removing the risk of losing in subsequent periods of the auctions. The existing models have invoked specific assumptions on bidders’ risk preferences—notably, that bidders exhibit NDARA in McAfee and Vincent (1993), or that bidders are averse solely to the price risk they pay in Mezzetti (2011).\textsuperscript{4} This paper extends the existing models to the more general environment and provides clear-cut predictions in terms of the expected price trends as a result of bidder risk preferences (Proposition 3). It shows that the price trend should decline over the auction periods if bidders are risk averse, and increase if bidders are risk-preferring.\textsuperscript{5} A particular finding of interest in

\textsuperscript{3}For example, see Ashenfelter (1989), Ashenfelter and Genesove (1992); Beggs and Graddy (1997); McAfee and Vincent (1993); Van den Berg, Van Ours and Pradhan (2001); among others.

\textsuperscript{4}Mezzetti (2011) obtained the declining price result for the case of private values. He also considered affiliated values and a non-standard formulation of English auctions, with mixed results on price trends.

\textsuperscript{5}Risk preferring preferences can be due to bidders financing their bids with borrowed money, or firms with existing debt that act to maximize the equity values of shareholders. Then, limited liability could lead to a convex payoff function and cause a bidder to behave like he is risk-preferring (see, e.g., the single-unit auction models of Zheng, 2001; DeMarzo, Kremer and Skrzypacz, 2005; Board, 2007).

\textsuperscript{6}Increasing price patterns are observed empirically in, e.g., Delas and Kosmopoulou (2004); Chanel et al. (1996); Jones et al. (2004); and Gandal (1997).
this study is that the declining price phenomena can be consistent with the generally accepted assumption that bidders exhibit NIARA preferences (see Section 6).

The rest of the paper is organized as follows. In Section 2, we present the model and the assumption of log-submodularity on the bidders’ marginal utilities. A number of special cases that are of interest are shown to be consistent with this assumption, including McAfee and Vincent (1993) and Mezzetti (2011) for his private values case. We then analyze the sequential Dutch auctions in Section 3, showing that it is ex post efficient as a consequence of the existence of a unique pure strategy equilibrium. Section 4 derives and characterizes the unique pure strategy symmetric equilibrium for the sequential Vickrey auctions, arriving at the same conclusion that the auctions are ex post efficient. Section 5 shows how bidders’ risk attitudes predict the behavior of price sequences. Section 6 discusses the background risk and explains why a marginal utility function that satisfies log-submodularity can derive from a primitive utility function that exhibits NIARA. Section 7 concludes the paper with some remarks on future research. Appendix A presents the useful properties that are implications of log-submodular marginal utilities, and Appendix B contains the proofs of the propositions.

2 Environment

A number $m \geq 1$ of identical objects are for sale sequentially, one at a time through periods $1, \ldots, m$ using either an open descending Dutch auction throughout, or a Vickrey auction throughout. There are $n > m$ competing bidders at the start, each having a single-unit demand for the object.

In each period $k$ of the Dutch auction, the price of an item steadily declines from a very high level until one of the bidders indicates that he is willing to pay. The bidder then becomes the winner of the $k$th auction and purchases the object at his stopping price. Because of the open format of the Dutch auction, the winning price in each period is publicly observed. The Dutch auction is thus strategically
equivalent to a first-price sealed-bid auction with the announcement of the winning price in each period.

In each period \(k\) of the Vickrey auction (or, equivalently, the second-price sealed-bid auction), the active bidders submit sealed bids and the highest bidder wins and pays the price equal to the second highest bid. Because of the sealed-bid policy, we assume that the remaining active bidders do not know the winning prices. However, in deriving the equilibrium strategies we will first assume that the winning bid is announced in each period. We then show that the announcement of the winning bid has no effect on the bidding strategies. This approach is standard in the literature that simplifies the equilibrium analysis.

In both Dutch and Vickrey auctions, we assume that if several bidders are tied in any period \(k\) with the same highest bid, then all of them will buy the remaining objects at their bid. If the remaining number of objects falls short of the number of the tied bidders, then the allocation will be resolved randomly, and the auction concludes. The reserve price in each period is normalized to be zero, and if a bidder does not win, he pays zero.

Each bidder \(i\) has a private type \(t_i \in [0, 1]\) that affects his preference for the object. Ex ante, the types \(t_i\) are independently distributed according to the same cumulative distribution \(F\) with \(F(0) > 0\) and density \(f = F'\) that is strictly positive and continuous on \([0, 1]\).

The preference of a typical bidder with type \(t\) is represented by

\[
\begin{cases} 
  w(x, t) & \text{if he wins the object and receives income } x \\
  u(x, t) & \text{if he loses and receives income } x
\end{cases}
\]

(1)

We interpret \(w : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}\) and \(u : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}\) as the bidder’s (induced) utilities conditional on winning and losing, respectively. In particular,

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7This is a mild assumption consistent with situations where the seller has a reserve price that is (slightly) higher than the lowest willingness-to-pay of the bidder types (see, e.g., Maskin and Riley, 2003). See also footnote 11.
$u(\cdot, t)$ is type-$t$ bidder’s status-quo utility for income, and a losing bidder after all objects are sold will have a utility denoted by $u(0, t)$. The preference model in (1) generalizes Maskin and Riley (1984) by allowing the private type $t$ of a bidder to matter in both winning and losing events. This generalization is particularly useful for incorporating background risks in our analysis.

We assume that $u$ and $w$ are twice continuously differentiable. In addition, the following assumptions will be maintained throughout the paper.

**Assumption 1** The partial derivatives $w_1(x, t) > 0$ and $w_2(x, t) > u_2(0, t)$ for all $x$ and $t$ such that $w(x, t) \geq u(0, t)$.

**Assumption 2** $w_1(x, t)$ is log-submodular in $(x, t)$ for all $x$ and $t$ such that $w(x, t) \geq u(0, t)$.

Assumption 1 provides the standard condition that utility increases in income, and a higher type makes a winning bidder better off as long as he prefers winning over losing. It is important to point out that no restriction is made on the signs of the partial derivatives $u_2(0, t)$ and $w_2(x, t)$.

A positive bivariate function $h(x, y)$ is log-submodular (log-supermodular) in $(x, y)$ if and only if for all $x < x'$ and $y < y'$ (see, e.g., Topkis, 1978),

$$h(x', y')h(x, y) \leq [\geq] h(x', y) h(x, y')$$

Therefore, Assumption 2 is equivalent to the assumption that $w_1(-p, t)$ is log-supermodular in $(p, t)$. Obviously, this assumption holds trivially true if $w$ is quasilinear in $x$. For a general function $w$, we summarize in Appendix A some of its useful properties when its partial derivative $w_1(x, t)$ is log-submodular.

For ease of exposition, the following lemma provides three equivalent statements in terms of log-submodular/log-supermodular marginal utilities and nonincreasing/nondecreasing absolute risk aversion of a von Neumann-Morgenstern utility function. The proof of the lemma amounts to straightforward verifications and is thus omitted (see, e.g., Athey, 2001 for part (iii) of the lemma).
Lemma 1 Let $U$ be a von Neumann-Morgenstern utility function with $U' > 0$. Then the following three conditions are equivalent:

(i) $U(x)$ exhibits nonincreasing [nondecreasing] absolute risk aversion.$^8$

(ii) $U'(x - y)$ is log-submodular [log-supermodular] in $(x, y)$.

(iii) $U'(x + y)$ or $U'(-x - y)$ is log-supermodular [log-submodular] in $(x, y)$.

Our environment incorporates many special cases of interest. For example, consider the following four.

Case 1. $u(0,t) \equiv U(0)$ and $w(x,t) = U(v(t) + x)$, with $v'(t) > 0$.

This is the standard private values model that has been extensively studied, including McAfee and Vincent (1993). Because $v' > 0$, by Lemma $1 w_1(x,t)$ is log-submodular if and only if $U$ exhibits NDARA, a condition required by McAfee and Vincent for the existence of a pure strategy symmetric equilibrium in their sequential first-price and second-price auctions.

Case 2. $u(0,t) \equiv U(0)$ and $w(x,t) = U(v(t) + \varphi(x))$, with $v'(t) > 0$ and $\varphi'(x) > 0$.

In this case, the object is of certain quality $v(t)$ that contributes to the utility. But the object may not have an equivalent monetary value. This is Case 2 of Maskin and Riley (1984). For $U$ risk neutral, define $\ell(p) = -\varphi(-p)$ and assume $\ell'' = -\varphi'' \geq 0$. Then $w(-p,t) = v(t) - \ell(p)$ and the model reduces to Mezzetti (2011) for his private-values case. The partial derivative $w_1(-p,t) = \ell'(-p)$ is independent of $t$, so that $[2]$ holds as an equality. This shows that Mezzetti’s private-values model satisfies Assumption $[2]$ and is therefore nested as a special case of our environment. For $U$ nonlinear, Assumption $[2]$ continues to hold for the NDARA class of functions $U$.

Case 3. $u(0,t) \equiv 0$ and $w(x,t) = \int \max(v + x - B, 0)dQ(v|t)$, where a higher $t$ shifts $Q$ to the right in the sense of first-order stochastic dominance.

$^8$That is, without restricting to risk aversion.
This case captures the effect of limited liability, where $B$ can be interpreted as the bidder’s liability or face value of debt. Because $w$ is now convex in $x$, we have $w_{11} > 0$ so the bidder’s induced utility $w$ is risk preferring. Suppose the density $Q_1(v|t)$ exists and is positive on the support of $v$. Then it can be readily verified that Assumption 2 holds if the hazard rate $(1 - Q(v|t)/Q_1(v|t)$ is nondecreasing in $t$ (e.g., Board, 2007).

Case 4. A bidder’s income $v$ has a distribution $K(v|t)$ if losing and $H(v|t)$ if winning, i.e.,

$$u(0,t) = \int_{-\infty}^{\infty} U(v) dK(v|t)$$
$$w(x,t) = \int_{-\infty}^{\infty} U(v + x) dH(v|t) \quad \text{with} \quad H(v|t) < K(v|t)$$

In this case winning allows the bidder to realize a more favorable income distribution $H(v|t)$, which dominates his status-quo income distribution $K(v|t)$ in the sense of first-order stochastic dominance. Broadly speaking, a bidder is exposed to background risk if $u(0,t)$ cannot be “normalized” as zero without losing generality. So Case 4 allows bidders to have exposures to both ensuing risk, since $v$ remains uncertain to the winner, and background risk. We will take a closer look at this case in Section 6.

It is worth remarking that the preference model in (1) can also accommodate non-expected utility preferences.

3 Sequential Dutch Auctions

We first look at the Dutch auctions. At the start, a bidding strategy for a bidder with type $t$ is a collection of $m$ bid functions $b_1, \ldots, b_m$ where $b_k(t|p_1, \ldots, p_{k-1})$ denotes his bid in the $k$th auction, given that he has lost the previous $k-1$ auctions and observed the winning prices $p_1, \ldots, p_{k-1}$. We focus on symmetric pure strategy equilibria in which $b_k$ is a continuous and increasing function of the bidder’s type $t$. The collection
of strategies \( \{b_k, k = 1, ..., m\} \) is a symmetric equilibrium of the sequential auctions game if in any period \( k \), every active bidder finds it optimal to play \( b_k \)—given that the other active bidders play strategy \( b_k \), and that all bidders, including the bidder himself, plan to play the remaining strategies \( \{b_\ell, \ell = k + 1, ..., m\} \) upon losing the \( k \)th auction.

For \( b_k \) continuous and increasing in \( t \), as will be verified, in equilibrium the winning bidder’s type in each auction is revealed to the remaining active bidders. By symmetry, w.l.o.g. we focus on analyzing the optimal strategies of bidder 1. Let the random variable \( Y_k \) denote the \( k \)th highest type from among the \( n - 1 \) bidders other than bidder 1, so that if bidder 1 with type \( t \) wins the \( k \)th auction, in equilibrium it must be the case that

\[
Y_k < t < Y_{k-1}, \quad k = 1, ..., m
\]

where \( Y_0 = \infty \) (by default). We let \( F_k(\cdot|y_{k-1}) \) denote the cumulative distribution, and \( f_k(\cdot|y_{k-1}) \) the associated density function, of \( Y_k \) conditional on \( Y_{k-1} = y_{k-1} \). So, the conditional equilibrium expected payoff for bidder 1 when he lost the previous \( k - 1 \) auctions and observed \( y_{k-1} \), can be specified recursively for all \( k \) by

\[
W^k_I(t|y_{k-1}) := w(-b_k(t), t)F_k(t|y_{k-1}) + \int_t^{y_{k-1}} W^{k+1}_I(t|y)dF_k(y|y_{k-1}) \tag{3}

\]

In (3), the first term on the right-hand side is associated with the winning event \( Y_k < t < y_{k-1} \), and the last term the losing event \( t < Y_k < y_{k-1} \). If the bidder loses in period \( k < m \), he still has the chance to win in the subsequent period \( k + 1 \) and hence attain the expected utility of \( W^{k+1}_I \). The final period expected payoff is given by

\[
W^m_I(t|y_{m-1}) = w(-b_m(t), t)F_m(t|y_{m-1}) + w(0, t)(1 - F_m(t|y_{m-1})) \tag{4}
\]

where, for \( m = 1 \), the equation reduces to the familiar specification of expected utility in single-unit first-price auctions.

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\(^9\)To ease analysis, we ignore the zero-probability events of ties, which will not affect the results of this paper.
Proposition 1  Under Assumptions 1 and 2, there exists a unique continuous and increasing symmetric equilibrium of the Dutch sequential auctions \( \{b_k : k = 1, ..., m\} \) characterized by

\[
\begin{align*}
    b'_m(t) &= (n - m) \frac{w(-b_m(t), t) - u(0, t)}{w_1(-b_m(t), t) F(t)} \\
    b'_k(t) &= (n - k) \frac{w(-b_k(t), t) - w(-b_{k+1}(t), t)}{w_1(-b_k(t), t) F(t)}, \quad k = 1, ..., m - 1
\end{align*}
\]

with the initial conditions \( b_k(0) = b_0 \) that solves \( w(-b_0, 0) = u(0, 0) \).

The differential equation (5) is similar to the standard characterization of equilibrium in the single-unit first-price auctions. For Cases 1–4 and \( m = 1 \), it can be readily verified that Assumption 2 is equivalent to that \( U(x) \) is log-concave. Therefore the characterization of \( b_m \) in (5) can be seen as a generalization of the existing symmetric first-price private values auctions in terms of the more general utility function \( w \) (e.g., Holt, 1980; Milgrom, 2004, Chapter 4.3). The differential equations in (6) extends McAfee and Vincent (1993) to the \( m \)-period environment, and Mezzetti (2011) to the more general risk preferences of the bidders.

In general, consistent with the existing results under risk neutrality (e.g., Milgrom and Weber, 2000, Krishna, 2010, Chapter 15), active bidders submit increasingly higher bids, i.e., for any \( t \) the bid \( b_k(t) \) increases as \( k \) increases. Another noteworthy point, as can be seen from (5)-(6), is that the previous winning prices have no influence on the remaining active bidders’ strategies.

In Proposition 1, Assumptions 1 and 2 are sufficient conditions that ensure the existence of a pure strategy equilibrium for arbitrary distribution functions \( F \) in the general environment. If the result is to hold as such in general, then Assumptions 1,2 are also necessary. For instance, McAfee and Vincent (1993) showed that for Case 1, if \( U \) exhibits decreasing absolute risk aversion (DARA), or equivalently, if Assumption 2 is violated, then there may not exist a pure strategy equilibrium.

An important consequence of Proposition 1 is that the sequential Dutch auction is ex post efficient in this environment. Essential for this result is the existence
of an increasing pure strategy equilibrium, which implies that all $m$ winners have higher willingness to pay than any of the losers do. In other words, when the auction concludes, no trade among bidders could lead to a Pareto superior re-allocation of the objects.

4 Sequential Vickrey Auctions

We analyze in this section the sequential Vickrey auction or its strategically equivalent format the sequential second-price sealed-bid auction. In each period, the winning bidder pays the price of the second highest bid, and the losers pay nothing. Let $\{a_k : k = 1, ..., m\}$ denote the collection of symmetric bid functions, and assume (and verify later) that each $a_k$ is a continuous and increasing function of type $t$. Following the literature, we assume that the winning bid, but not the winning price, is announced in each auction period. As it turns out, however, similar to the Dutch auctions, in equilibrium the knowledge of the winning bid in each period has no effect on the active bidders' remaining strategies.

The equilibrium expected payoff for bidder 1 if he lost the previous $k - 1$ auctions and observed $Y_{k-1} = y_{k-1}$, can be specified recursively for all $k$ by

$$W^k_{II}(t|y_{k-1}) := \int_0^t w(-a_k(y), t)dF_k(y|y_{k-1}) + \int_t^{y_{k-1}} W^{k+1}_{II}(t|y)dF_k(y|y_{k-1}),$$

where the first term on the right-hand side is associated with the winning event $Y_k < t < y_{k-1}$ in period $k$, and the last term the event $t < Y_k < y_{k-1}$, with the final period expected payoff given by

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10If the winning price is announced in each period, the situation resembles a sequence of open ascending English auctions. As some of the pivotal or highest losing bidders would still be active, if their bids were known the existence of a pure strategy equilibrium can be difficult to establish. The conventional model of the button-English auctions may become inappropriate as well. See, e.g., Mezzetti (2011).
\[ W_m^t (t|y_{m-1}) = \int_0^t w(-a_m(y), t) dF_m(y|y_{m-1}) + u(0, t)(1 - F_m(t|y_{m-1})) \]

**Proposition 2** Under Assumptions 1 and 2, there exists a unique continuous and increasing symmetric equilibrium of the Vickrey sequential auctions \( \{a_k : k = 1, ..., m \} \) satisfying

\[ w(-a_m(t), t) = u(0, t) \] (7)

\[ w(-a_k(t), t) = \int_0^t w(-a_{k+1}(y), t) dF_{k+1}(y|t) \] (8)

The proposition shows that in every period \( k \), the previous winner’s type \( y_{k-1} \) does not appear in (8) so that any realization of \( Y_{k-1} \) has no influence on the subsequent equilibrium bids. Instead, the bidder calculates his expected subsequent period payoff using the distribution \( F_{k+1}(y|t) \) of \( Y_{k+1} \) conditional on the event \( Y_k = t \).

The equation in (7) of the proposition provides the familiar (weakly) dominant strategy of bidding up to one’s break-even level, when there is a single object left for sale. The equations (8) say about the same thing, although not in terms of dominant strategies: in the sequential Vickrey auction it is optimal to bid up to the level in each period \( k \) such that the bidder is indifferent whether paying his bid and win or losing the \( k \)th auction. This reveals an interesting link between the private-values sequential Vickrey auction, as modelled here, and the single-unit Vickrey auction with affiliated values, as modelled in Milgrom and Weber (1982). In both models, it is optimal for a bidder to bid up to an amount such that he will be at tie with the next potential winner, i.e., \( Y_k = t \).

In light of McAfee and Vincent’s observation (1993, Remark 3), for Case 1, that NDARA is necessary for the existence of a pure strategy equilibrium in a two-unit second-price sequential auction, Assumption 2 can also be seen, to some extent, necessary for the result of Proposition 2.

Like the sequential Dutch auction, the pure strategy equilibrium of the sequential Vickrey auction is ex post Pareto efficient. This conclusion, however, holds for
the symmetric equilibrium characterized in Proposition 2. It is well-known that the Vickrey auctions can have asymmetric equilibria that may not be ex post efficient.

5 Price Trends

The equilibrium strategies derived in Propositions 1 and 2 have clear-cut implications for the expected price trends as consequences of bidders’ risk preferences.

**Proposition 3** Under Assumptions 1 and 2, let \( p_1, \ldots, p_m \) be the prices that the objects are sold in periods \( 1, \ldots, m \) of the Dutch or Vickrey auctions, respectively. Then, for all \( k = 1, \ldots, m - 1 \),

1. if \( w_{11} < 0 \), then \( E(\tilde{p}_{k+1}|p_k) < p_k \);
2. if \( w_{11} = 0 \), then \( E(\tilde{p}_{k+1}|p_k) = p_k \);
3. if \( w_{11} > 0 \), then \( E(\tilde{p}_{k+1}|p_k) > p_k \).

The intuition of this proposition is that for risk averse bidders, the risk of losing outweighs the opportunity of winning in subsequent periods of the auction. Consequently, they are more eager to avoid the risk of losing and are thus willing to pay a risk premium for it. For risk preferring bidders, they value more the option of waiting and winning in subsequent periods, and are thus reluctant to bid too high in earlier rounds.

6 Background Risk

As mentioned in the introduction, bidders in sequential auctions are conceivably more likely to be firms or individuals who have a business to run. A bidder’s objective of acquiring the auctioned object can be just to maintain an ongoing business. Under these circumstances, participating in an auction can be motivated by either seeking potential profits or avoiding potential losses or both. For instance, losing in the auctions could mean losses of sales, customers, or any kind of unintended
consequence that a bidder would like to avoid. Such background risk, as we call it, is perhaps inconsequential when bidders are risk neutral. In this case only the difference between winning and losing matters so that the status-quo utility of a bidder may be normalized as zero. However, when bidders are not risk neutral, such a normalization removes an important part of reality—that is, when winning helps reduce the background risks. Indeed, taking background risks into consideration significantly enlarges the scope of our theoretical predictions. In particular, it allows us to explain the declining price anomaly in sequential auctions under the common assumption that bidders exhibit NIARA.

To gain some insight into this conclusion, let us compare the following two simple models where $p$ denotes the price paid for the object upon winning, and $U$ is a utility function with $U' > 0$.

**Model 1:** $u(0, t) = U(0)$ and $w(-p, t) = U(-p + t)$.

This is the standard private values model of Case 1, where a higher $t$ indicates a higher value to the bidder upon winning.

**Model 2:** $u(0, t) = U(-t)$ and $w(-p, t) = U(-p)$.

This is a pure background risk model where a higher $t$ indicates a higher level of loss to the bidder upon losing.

For $U$ risk neutral, both models imply $w(-p, t) - u(0, t) = t - p$ so that the two models coincide. In general, $U' > 0$ ensures that Assumption 1 holds in both models. By Lemma 1 however, Assumption 2 holds in Model 1 only if $U$ exhibits NDARA, but it holds trivially true for arbitrary utility functions $U$ in Model 2.

Now to give Model 2 some practical interpretations, consider a more general situation of “bidding to fill the orders.” Suppose the losing bidder has a status-quo utility $u(0, t) = U(-C(t))$, where $C$ is a cost function that increases in $t$, e.g., cost of losing sales or customers due to failure of delivering the orders. Suppose winning removes (part of) that potential cost, generating a utility of $w(-p, t) = U(-p - \theta C(t))$ to the bidder for some $\theta \in [0, 1)$. The parameter $\theta$ measures the
The effectiveness of the auctioned object in reducing the bidder’s background risk. Then, by Lemma \[1\] Assumption \[2\] holds if and only if \( U \) exhibits NIARA.

The above examples are only special, degenerate, instances of the general model of Case 4. The following corollary presents the main result of this section.

**Corollary 1** Let \( U \) be a von Neumann-Morgenstern utility function with \( U' > 0 \), and suppose

\[
\begin{align*}
  u(0,t) &= \int_{-\infty}^{\infty} U(v) dK(v|t) \\
  w(x,t) &= \int_{-\infty}^{\infty} U(v+x) dH(v|t) \text{ with } H(v|t) < K(v|t)
\end{align*}
\]

such that \( w_2(x,t) > u_2(0,t) \) as long as \( w(x,t) \geq u(0,t) \). Further suppose that \( U \) exhibits NIARA and the density \( H_1(v|t) \) is log-submodular in \((v,t)\). Then there exists a pure strategy equilibrium in the sequential Dutch and Vickrey auctions as characterized in Proposition \[1\] and Proposition \[2\] respectively.

**Proof.** It suffices to show that \( w_1(x,t) \) is log-submodular so that Assumption \[2\] holds. To see this, let \( y = -v \) and \( p = -x \). Then \( H_1(-y|t) \) is log-supermodular in \((y,t)\) by assumption that \( H_1(v|t) \) is log-submodular in \((v,t)\). By Lemma \[1\] a NIARA utility function \( U \) has the property that \( U'(-y - p) \) is log-supermodular in \((p,y)\). Consequently,

\[
w_1(-p,t) = \int_{-\infty}^{\infty} U'(v-p)H_1(v|t) dv = \int_{-\infty}^{\infty} U'(-y - p)H_1(-y|t)dy
\]

is log-supermodular in \((p,t)\), or equivalently, \( w_1(x,t) \) is log-submodular in \((x,t)\). This derives from the fact that log-supermodularity is preserved under integration (see, e.g., Athey, 2002). □

This corollary extends and clarifies the conclusion of McAfee and Vincent (1993) that NDARA preferences are necessary for the existence of pure strategy equilibria in sequential auctions. The corollary suggests that there are plausible circumstances in which NIARA, rather than NDARA, leads to a pure strategy
equilibrium. It can be readily verified that the assumptions of the corollary imply $K_2(v|t) > 0$, so that $u(0,t)$ is a decreasing function of $t$. This means that a higher type implies higher losses in case of losing in the auctions so that a bidder’s willingness-to-pay is directly related to the severity of his background risk.

7 Concluding Remarks

This paper establishes the existence of a continuous and increasing pure strategy equilibrium in the sequential Dutch (or first-price sealed-bid) auction, and in the sequential Vickrey (or second-price sealed-bid) auction, respectively. We obtain these results under a general environment where bidders can have non-quasilinear utilities as well as heterogeneous exposures to risk, and there can be an arbitrary number of identical objects for sale with unit demand. Because the winners have in general higher willingness-to-pay than the losers, a desirable implication of these existence results is that sequential auctions are inductive to ex post Pareto efficiency. Thus, it provides a justification for the prevalence of sequential auctions in practice.

A key assumption underlying the existence of pure strategy equilibria in sequential auctions is that bidders’ marginal utilities exhibit log-submodularity in income and type, or equivalently, log-supermodularity in payment and type. The special cases (Cases 1-4) demonstrate that this condition is fairly general. By taking the log-submodular/log-supermodular approach, we are able to provide the clear characterizations of the bidding strategies in Propositions 1 and 2. The simplified analysis also allows us to obtain sharp predictions about the price trends in sequential auctions in Proposition 3. Specifically, we show that risk averse (preferring) bidders have the expectations of decreasing (increasing) price sequences in equilibrium. A subset of the general environment considered in this paper allows bidders to have exposures to the background risk. As shown in Section 6, in these situations the “declining price anomaly” is, in fact, not an anomaly but a natural consequence of nonincreasing absolute risk aversion.
We have restricted attention to the information structure involving “private values” in that a bidder’s willingness-to-pay is unaffected by the other bidders’ private information. Conceivably, efficiency may be harder to achieve if bidders have interdependent willingness-to-pay and affiliated types or signals (see, e.g., Hu, Matthews and Zou, 2014, Example 2). Conditions that may ensure ex post efficiency under these more general information structures are certainly worthy of investigations in future research. The relative strengths and weaknesses of different sequential auction policies may be also investigated in the more general informational setting, in terms of expected revenue and Pareto efficiency. Given the insights derived from the present study, we surmise that the key to finding and solving pure strategy equilibria in the more general settings still hinges upon the assumption of log-submodular marginal utilities.

Appendix A. Implications of Assumption 2

In this appendix we summarize five useful properties that are implications of the log-submodular marginal utilities. Related results in terms of the log-supermodular marginal utilities can be found in Hu, Matthews and Zou (2014).

**Property 1.** Assumptions 1 and 2 imply

\[ \frac{w(x,t) - w(y,t)}{w_1(x,t)} \leq \frac{w(x,t') - w(y,t')}{w_1(x,t')}, \forall t < t' \text{ and } \forall x, y \]  \hspace{1cm} (9)

This derives from (2) straightforwardly by integrations. Note that in (9), the signs of \( x, y, \) and \( x - y \) are arbitrary.

**Property 2.** Assumptions 1 and 2 imply that for all \( t < t' \), and \( x \) such that \( w(x,t) > u(0,t) \),

\[ \frac{w(x,t) - u(0,t)}{w_1(x,t)} \leq \frac{w(x,t') - u(0,t')}{w_1(x,t')}, \forall t < t' \]  \hspace{1cm} (10)

To see this, define \( a(t) \) by \( w(-a(t), t) = u(0,t) \). Then (10) is equivalent to

\[ \frac{w(x,t) - w(-a(t),t)}{w_1(x,t)} \leq \frac{w(x,t') - w(-a(t'),t')}{w_1(x,t')}, \forall t < t' \]
By Assumptions 1,

\[ a'(t) = \frac{w_2(-a(t), t) - w_2(0, t)}{w_1(-a(t), t)} > 0 \]

So (9) implies that \( (w(x, t) - w(-a(t), t)) / w_1(x, t) \) is an increasing function of \( t \).

**Property 3.** Property 1 implies that for all random \( \tilde{y} \) such that \( E w(\tilde{y}, t) \) exists (replace \( y \) by \( \tilde{y} \) and take expectation over \( \tilde{y} \) in (9)),

\[ \frac{w(x, t) - E w(\tilde{y}, t)}{w_1(x, t)} \leq \frac{w(x, t') - E w(\tilde{y}, t')}{w_1(x, t')}, \forall t < t' \text{ and } \forall x \] (11)

Therefore, (9) has an economic interpretation that increasing \( t \) makes the utility function \( w(x, t) \) (weakly) more risk averse in income \( x \).

**Property 4.** Suppose (9) holds, then

\[ w(x, t) = E w(\tilde{y}, t) \text{ implies } w_2(x, t) \geq E w_2(\tilde{y}, t) \] (12)

This result is well known (e.g., McAfee and Vincent, 1993 and the references therein).

**Property 5.** Assumptions 1 and 2 imply

\[ \frac{w(x, t) - w(y, t)}{w_1(x, t)} \begin{cases} > x - y & \text{if } w_{11} < 0 \\ = x - y & \text{if } w_{11} = 0 \\ < x - y & \text{if } w_{11} > 0 \end{cases} \] (13)

This can be seen by letting \( w(x, t_0) \) denote a risk neutral utility function in \( x \), so that \( w(x, t_0) = h(t_0)x + g(t_0) \). Then, (13) follows as special cases of (9) because (11) implies \( w_{11}(\cdot, t) \leq [\geq] 0 \text{ iff } t \geq [\leq] t_0 \).

**Appendix B. Proofs of Propositions**

**Proof of Proposition 1.** We analyze by backward induction bidder 1’s optimal response, assuming that all others play a given set of continuous and increasing strategies \( \{b_k : k = 1, ..., m\} \). For notational convenience, we write \( b_k(t) \) rather than
Taking partial derivative w.r.t. though his type was \( Y \) observed for each \( b \) the single-crossing condition of Milgrom and Shannon (1994) and the existence of therefore \( V \) there are two possibilities: As we do not assume that bidder \( b \) do not influence the bid function \( b_k \). We take the standard approach by first assuming for each \( k \) that \( b'_k > 0 \), and then verify that this is indeed true upon establishing its existence.

In the last period, suppose bidder 1 has lost the preceding \( m - 1 \) auctions and observed \( Y_{m-1} = y_{m-1} \). The bidder’s expected payoff if he has type \( t \) and bids as though his type was \( z \) equals

\[
V^m(z, t|y_{m-1}) := w(-b_m(z), t) F_m(z|y_{m-1}) + u(0, t)(1 - F_m(z|y_{m-1})) \tag{14}
\]

Taking partial derivative w.r.t. \( z \) gives

\[
V_1^m(z, t|y_{m-1}) = -b'_m(z) w_1(-b_m(z), t) F_m(z|y_{m-1}) + \left( w(-b_m(z), t) - u(0, t) \right) f_m(z|y_{m-1}) \]

\[
= w_1(-b_m(z), t) F_m(z|y_{m-1}) \left( \frac{w(-b_m(z), t) - u(0, t)}{w_1(-b_m(z), t)} \frac{f_m(z|y_{m-1})}{F_m(z|y_{m-1})} - b'_m(z) \right) \tag{15}
\]

As we do not assume that bidder 1 has followed the equilibrium strategies previously, there are two possibilities: \( t < y_{m-1} \) and \( t > y_{m-1} \). We first assume \( t < y_{m-1} \), which is a consequence of equilibrium play given increasing bid functions \( b_k \). Since \( b'_m > 0 \), from (15) it can be seen that if \( V_1^m(z, t|y_{m-1}) \geq 0 \) then \( w(-b_m(z), t) > u(0, t) \). By (10) in Appendix A, \( V_1^m(z, t|y_{m-1}) \geq 0 \) thus implies

\[
\frac{w(-b_m(z), t) - u(0, t)}{w_1(-b_m(z), t)} \leq \frac{w(-b_m(z), t') - u(0, t')}{w_1(-b_m(z), t')} \text{ for all } t' > t
\]

and therefore \( V_1^m(z, t'|y_{m-1}) \geq 0 \) for all \( t' > t \). Consequently, \( V^m(z, t|y_{m-1}) \) satisfies the single-crossing condition of Milgrom and Shannon (1994) and the existence of \( b_m \) is guaranteed under the assumption that \( b_m \) is nondecreasing (e.g., Athey, 2001).

From (15), \( V_1^m(t, t|y_{m-1}) = 0 \) implies (5):

\[
b'_m(t) = \frac{w(-b_m(t), t) - u(0, t) f_m(t|y_{m-1})}{w_1(-b_m(t), t) F_m(t|y_{m-1})} = \frac{(n - m) w(-b_m(t), t) - u(0, t) f(t)}{w_1(-b_m(t), t) F(t)}
\]

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where the last equation derives from
\[ F_k(x|y_{k-1}) = \frac{F(x)^n - k}{F(y_{k-1})^{n-k}} \]  
(16)

As for the case \( t > y_{m-1} \), it is only possible if in the preceding auction the bidder has deviated from the equilibrium strategy and bid higher than \( b_{m-1}(t) \). Given \( y_m < y_{m-1} < t \), it does not make sense to bid above \( b_m(y_{m-1}) \). In this case \( V^m_1(y_{m-1}, t|y_{m-1}) \geq 0 \) and so it is optimal for the bidder to bid \( b_m(y_{m-1}) \) and win the \( m \)th auction with certainty.

We now prove (6) by backward induction on \( k \). For \( k = m \), we have derived that given any \( y_{m-1} \),
\[ V^m(t \wedge y_{m-1}, t|y_{m-1}) = \max_z V^m(z, t|y_{m-1}) \]  
(17)

where \( t \wedge y_k = \min(t, y_k) \). Now suppose the bidder has lost the previous auctions up to the \( k \)th auction for \( k \leq m - 1 \), and observes \( Y_{k-1} = y_{k-1} \). Our induction hypothesis is
\[ V^{k+1}(t \wedge y_k, t|y_k) = \max_z V^{k+1}(z, t|y_k), \ \forall y_k \]  
(18)

for the subsequent auctions \( k + 1, \ldots, m \). That is, if the bidder loses the \( k \)th auction and observes \( Y_k = y_k \), sequential rationality calls on him to bid \( b_{k+1}(t \wedge y_k) \) in the \((k+1)\)th auction and so on. Hence, if the bidder bids now as though his type was \( z \), his expected payoff equals
\[ V^k(z, t|y_{k-1}) := w(-b_k(z), t)F_k(z|y_{k-1}) + \int_z^{y_{k-1}} V^{k+1}(t \wedge y, t|y)F_k(y|y_{k-1}) \]  
(19)

Differentiating yields
\[ V^k_1(z, t|y_{k-1}) \]
\[ = -b'_k(z)w_1(-b_k(z), t)F_k(z|y_{k-1}) + \left( w(-b_k(z), t) - V^{k+1}(t \wedge z, t|z) \right)f_k(z|y_{k-1}) \]
\[ = w_1(-b_k(z), t)F_k(z|y_{k-1}) \]
\[ \times \left( \frac{w(-b_k(z), t) - V^{k+1}(t \wedge z, t|z)F_k(z|y_{k-1})}{w_1(-b_k(z), t)F_k(z|y_{k-1}) - b'_k(z)} \right) \]  
(20)
Substituting $z$ for $y_{k-1}$ in (19) reveals
\[ V^k(z, t|z) = w(-b_k(z), t)F_k(z|z) = w(-b_k(z), t) \tag{21} \]
for all $k$. So for $z \leq t$, $V^k_1(z, t|y_{k-1}) \geq 0$ and $b'_k(z) > 0$ imply
\[ w(-b_k(z), t) > V^{k+1}(t \land z, t|z) = w(-b_{k+1}(z), t) \]
and consequently, by (21), and (9) in Appendix A,
\[ \frac{w(-b_k(z), t) - V^{k+1}(t \land z, t|z)}{w_1(-b_k(z), t)} \text{ is nondecreasing in } t \tag{22} \]
This shows that $V^k_1(z, t'|y_{k-1}) \geq 0$ for all $t' > t$. The single-crossing condition thus holds for $V^k(z, t|y_{k-1})$ for all $z \leq t$.

For $z > t$, we show $V^k_1(z, t|y_{k-1}) < 0$ and so it is never optimal to bid higher than the equilibrium play (which then of course implies the single-crossing condition). We have now
\[ V^{k+1}(t \land z, t|z) = V^{k+1}(t, t|z) > V^{k+1}(z, t|z) = w(-b_{k+1}(z), t), \forall z > t \]
where the inequality is due to (18), and the equalities are due to (21) and $t < z$.

Therefore, the term in large braces of (20)
\[
\begin{align*}
&\frac{w(-b_k(z), t) - V^{k+1}(t \land z, t|z)}{w_1(-b_k(z), t)} \frac{f_k(z|y_{k-1})}{F_k(z|y_{k-1})} - b'_k(z) \\
<& \frac{w(-b_k(z), t) - w(-b_{k+1}(z), t) f_k(z|y_{k-1})}{w_1(-b_k(z), t)} \frac{F_k(z|y_{k-1})}{F_k(z|y_{k-1})} - b'_k(z) \text{ by (9)} \\
&\leq \frac{w(-b_k(z), z) - w(-b_{k+1}(z), z) f_k(z|y_{k-1})}{w_1(-b_k(z), z)} \frac{F_k(z|y_{k-1})}{F_k(z|y_{k-1})} - b'_k(z) \\
&= 0 \text{ for all } z > t \text{ by (6)}
\end{align*}
\]
Bidding $b_k(t)$ is therefore optimal, yielding
\[ V^k(t, t|y_{k-1}) = \max_z V^k(z, t|y_{k-1}), \forall y_{k-1} \]

The first order condition $V^k_1(t, t|y_{k-1}) = 0$ now gives
\[ b'_k(t) = \frac{w(-b_k(t), t) - w(-b_{k+1}(t), t) f_k(t|y_{k-1})}{w_1(-b_k(t), t)} \frac{F_k(t|y_{k-1})}{F_k(t|y_{k-1})} \tag{23} \]
which, by (16), reduces to (6). Note that this also implies that for the subsequent auction, $V^{k+1}(t, t|y_k) = \max_z V^{k+1}(z, t|y_k), \forall y_{k-1}$. By the induction hypothesis, we conclude that $\{b_k : k = 1, \ldots, m\}$ constitute a symmetric equilibrium for the sequential Dutch auctions.

The uniqueness of the equilibrium can be established also by induction on $k$. By the fundamental theorem of ordinary differential equations, $b_m$ is unique under the initial condition $b_m(0) = b_0$ because the right-hand side of (5) is continuously differentiable in $b_m$ and continuous in $t$ on a compact set $[0, 1]$.[1] Now suppose $b_{k+1}$ is unique. Then because $b_{k+1}(t)$ is differentiable, the right-hand side of (6) is continuously differentiable in $b_k$ and continuous in $t \in [0, y_{k-1}]$ so that $b_k$ is unique with $b_k(0) = b_0$. By induction, the uniqueness of $b_k$ holds therefore for all $k = 1, \ldots, m$.

Finally, we verify that $b'_k > 0$ by backward induction. For $k = m$, let $A(t) = w(-b_m(t), t) - u(0, t)$ so that $b'_m(t) > 0$ iff $A(t) > 0$. We have $A(0) = 0$. For $t \geq 0$, if $A(t) = 0$ then $b'_m(t) = 0$ so that $A'(t) = w_2(-b_m(t), t) - u_2(0, t) > 0$ by Assumption [1]. This implies $A(t) > 0$ for all $t \in (0, 1]$ (e.g., Hu, Matthews and Zou, 2010, Lemma 1(i)). Now suppose $b'_{k+1}(t) > 0$ on $(0, 1]$ and we show $b'_k > 0$ on $(0, 1]$. Let $B(t) = b_{k+1}(t) - b_k(t)$ so that by (6), $b'_k(t) > 0$ iff $B(t) > 0$. We have $B(0) = 0$. For $t > 0$, if $B(t) \leq 0$ then $b'_k(t) \leq 0$ so that $B'(t) = b'_{k+1}(t) - b'_k(t) \geq b'_{k+1}(t) > 0$. This implies $B(t) > 0$ on $(0, 1]$ (e.g., Matthews and Zou, 2010, Lemma 1(ii)). So by induction, $b'_k > 0$ on $(0, 1]$ for all $k = 1, \ldots, m$. ■

**Proof of Proposition 2.** We prove again the existence and uniqueness of equilibrium by backward induction on $k$. In the last period $k = m$, it is a (weakly) dominant strategy for bidder 1 with type $t$ to bid $a_m(t)$ according to (7). Because $w_1$ is continuous and positive, $a_m(t)$ is uniquely defined, and by Assumption [1] it's

---

[1] This is where $F(0) > 0$ is useful for simplifying the uniqueness argument. If $F(0) = 0$, then the differential equation has a singular point at $t = 0$. But it still can be shown that the uniqueness holds in this case (e.g., see Maskin and Riley, 1984, Remark 2.1).
derivative is positive:

\[
a'_m(t) = \frac{w_2(-a_m(t), t) - u_2(0, t)}{w_1(-a_m(t), t)} > 0
\]  

(24)

Assuming all other active bidders play \(a_m\) as well, the expected payoff of bidder 1 at the start of the \(m\)th auction knowing \(Y_{m-1} = y_{m-1}\) equals

\[
V^m(t \land y_{m-1}, t|y_{m-1}) := \int_{0}^{t \land y_{m-1}} (w(-a_m(y), t) - u(0, t)) dF_m(y|y_{m-1}) + u(0, t)
\]

where the notation \(t \land y_{m-1}\) incorporates the possibility that bidder 1 has deviated from the equilibrium bid in the previous auction, resulting in \(y_{m-1} < t\).

Now consider the \(k\)th auction for \(k \leq m - 1\) assuming that a unique sequence of increasing strategies \(a_{k+1}, ..., a_m\) will be played by all bidders including bidder 1. Let \(V^{k+1}(z \land y_k, t|y_k)\) denote the bidder’s expected payoff if he bids in period \(k + 1\) as though his type was \(z\). Our induction hypothesis is that for all \(k \leq m - 1\),

\[
V^{k+1}(z \land y_k, t|y_k) = \int_{0}^{z} w(-a_{k+1}(y), t) - u(0, t) dF_{k+1}(y|y_k) + u(0, t)
\]

(25)

\[
\geq V^{k+1}(z \land y_k, t|y_k) \text{ for all realized } y_k \text{ in the } k\text{th auction and } z
\]

Suppose \(Y_{k-1} = y_{k-1}\) and bidder 1 is still active in the \(k\)th auction. Obviously, bidding above \(a_k(y_{k-1})\) is weakly dominated by bidding lower than or equal to \(a_k(y_{k-1})\).

So given that the other bidders play \(a_k\), if bidder 1 bids as though his type was \(z\), his expected payoff equals

\[
V^{k}(z, t|y_{k-1}) = \int_{0}^{z} w(-a_k(y), t)dF_k(y|y_{k-1}) + \int_{z}^{y_{k-1}} V^{k+1}(t \land y, t|y)dF_k(y|y_{k-1})
\]

(26)

Differentiating w.r.t. \(z\) gives

\[
V^k_1(z, t|y_{k-1})
= \left( w(-a_k(z), t) - V^{k+1}(t \land z, t|z) \right) f_k(z|y_{k-1})
= \left( w(-a_k(z), t) - u(0, t) - \int_{0}^{t \land z} (w(-a_{k+1}(y), t) - u(0, t)) dF_{k+1}(y|z) \right) f_k(z|y_{k-1})
\]
Equation (8) implies

$$w(-a_k(z), z) - u(0, z) = \int_{0}^{z} (w(-a_{k+1}(y), z) - u(0, z)) \, dF_{k+1}(y | z)$$  \hspace{1cm} (27)$$

If \( z < t \) then the term in large braces of (26) reduces to

$$w(-a_k(z), t) - \int_{0}^{z} w(-a_{k+1}(y), t) \, dF_{k+1}(y | z)$$

which, by (11) in Appendix A, is non-negative because

$$\frac{w(-a_k(z), t) - \int_{0}^{z} w(-a_{k+1}(y), t) \, dF_{k+1}(y | z)}{w_1(-a_k(z), t)} \geq \frac{w(-a_k(t), z) - \int_{0}^{z} w(-a_{k+1}(y), z) \, dF_{k+1}(y | z)}{w_1(-a_k(t), z)} = 0 \text{ by (8)}$$

If \( z > t \), then the term in large braces of (26)

$$w(-a_k(z), t) - u(0, t) - \int_{0}^{t} (w(-a_{k+1}(y), t) - u(0, t)) \, dF_{k+1}(y | z)$$

$$< w(-a_k(z), t) - \int_{0}^{z} w(-a_{k+1}(y), t) \, dF_{k+1}(y | z)$$

$$\leq 0 \text{ by (8)}$$

Consequently, \( V^k_1(t, t | y_{k-1}) = 0 \) is a necessary and globally sufficient condition for \( a_k(t) \) to be optimal for bidder 1. Since \( a_{k+1}, \ldots, a_m \) are unique, \( a_k(\cdot) \) is uniquely defined implicitly by (8) given \( w_1 > 0 \).

Finally, we show that \( a_k(\cdot) \) is increasing by backward induction. For \( k = m \) (24) shows \( a'_m > 0 \). Suppose \( a'_{k+1} > 0 \) and differentiate (8) to get

$$a'_k(t) = \frac{w_2(-a_k(t), t) - \int_{0}^{t} w_2(-a_{k+1}(y), t) \, dF_{k+1}(y | t)}{w_1(-a_k(t), t)}$$

$$+ (n - k) \frac{f(t)}{F(t)} \left( \frac{w(-a_k(t), t) - w(-a_{k+1}(t), t)}{w_1(-a_k(t), t)} \right)$$

By (8) and (12), the first term on the right-hand side is non-negative. Since \( a'_{k+1} > 0 \) as assumed, (8) implies \( a_k(t) < a_{k+1}(t) \). Hence the last term in the above expression is positive. The conclusion of the proposition is thus established by induction. ■
Proof of Proposition 3. Dutch auctions. We show that (i)-(iii) follow from (13) in Appendix A, because (6) and $F_k(x|y_{k-1}) = F(x)^{n-k}/F(y_{k-1})^{n-k}$ imply

$$b'_k(t) = \frac{w(-b_k(t), t) - w(-b_{k+1}(t), t)}{w_1(-b_k(t), t)} f_k(t|y_{k-1})$$

$$> \left[ \frac{b_{k+1}(t)}{b_k(t)} \right] b_k(t) F_k(t|y_{k-1})$$

if $w_{11} < 0$  \( \tag{28} \)

The risk neutral case (ii) is well known. Now we show Case (i) assuming that $w_{11} < 0$. Rearranging terms of the first inequality in (28) gives

$$\frac{d}{dt} [b_k(t) F_k(t|y_{k-1})] > b_{k+1}(t) f_k(t|y_{k-1})$$

For $t = y_{k+1}$, integrating over $[0, y_k]$ yields

$$b_k(y_k) > \frac{1}{F_k(y_k|y_{k-1})} \int_0^{y_k} b_{k+1}(y_{k+1}) dF_k(y_{k+1}|y_{k-1})$$

$$= \frac{1}{F^{n-k}(y_k)} \int_0^{y_k} b_{k+1}(y_{k+1}) dF^{n-k}(y_{k+1})$$

$$= E(b_{k+1}(Y_{k+1})|Y_{k+1} < y_k)$$

$$= E(\tilde{b}_{k+1}|b_k(y_k))$$

In the Dutch auction, $p_k = b_k(y_k)$. So we have shown that $w_{11} < 0$ implies $E(\tilde{p}_{k+1}|p_k) < p_k$. Case (iii) can be established by simply altering the sign of the inequality, and so is omitted.

Vickrey auctions. By (8), we have

$$w(-a_k(y_k), y_k) = \int_0^{y_k} w(-a_{k+1}(y), y_k) dF_{k+1}(y|y_k)$$

This shows that for $w(\cdot, y_k)$, the payment $a_k(y_k)$ is the certainty equivalent of random payment $a_{k+1}(y)$, under expectation according to distribution $F_{k+1}(y|y_k)$. Therefore,

$$a_k(y_k) = \begin{cases} 
> \int_0^{y_k} a_{k+1}(y) dF_{k+1}(y|y_k) & \text{for } w_{11} < 0 \\
= \int_0^{y_k} a_{k+1}(y) dF_{k+1}(y|y_k) & \text{for } w_{11} = 0 \\
< \int_0^{y_k} a_{k+1}(y) dF_{k+1}(y|y_k) & \text{for } w_{11} > 0 
\end{cases}$$
Since $a_k(y_k)$ and $y_k$ are one-to-one, we obtain

$$p_k = a_k(y_k) \begin{cases} > \int_{0}^{y_k} a_{k+1}(y)dF_{k+1}(y|y_k) = E(\tilde{p}_{k+1}|p_k) \text{ for } w_{11} < 0 \\ = \int_{0}^{y_k} a_{k+1}(y)dF_{k+1}(y|y_k) = E(\tilde{p}_{k+1}|p_k) \text{ for } w_{11} = 0 \\ < \int_{0}^{y_k} a_{k+1}(y)dF_{k+1}(y|y_k) = E(\tilde{p}_{k+1}|p_k) \text{ for } w_{11} > 0 \end{cases}$$

References


