Well-Conditioned Configurations of Fault-Tolerant Manipulators

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Abstract: Fault-tolerant motion of redundant manipulators can be obtained by joint velocity reconfiguration. For fault-tolerant manipulators, it is beneficial to determine the configurations that can tolerate the locked-joint failures with a minimum relative joint velocity jump. This is because the manipulator can rapidly reconfigure itself to tolerate the fault. This paper uses the properties of condition numbers to introduce those optimal configurations for serial manipulators. The relationship between the manipulator’s locked-joint failures and the condition number of the Jacobian matrix is indicated by using a matrix perturbation methodology. It is observed that the condition number provides the upper bound of required relative joint velocity change for recovering the faults which leads to define the optimal fault-tolerant configuration from the minimization of the condition number. The optimization problem to obtain the minimum condition number is converted to three standard Eigen value optimization problems. Finally, in order to obtain the optimal fault-tolerant configuration, the proposed method is applied to a 4-DoF planar manipulator.

Keywords: Redundant manipulator, Condition number, Optimal fault tolerance, Actuator failure, Isotropic manipulators.

I. INTRODUCTION

Fault-tolerant manipulators are essential where highly dependable robots are required such as robotic manipulators working in hazardous environments, nuclear disposal and exploring of deep sea or outer space [1-3]. The design or control of a fault-tolerant manipulator aims to maintain the dependability of the manipulator despite partial failures that can occur in the manipulator’s actuators or sensors [4, 5]. The literature surrounding fault-tolerant manipulators focuses on the design or control of the manipulators. Within the literature dealing with design of the manipulators, different structures such as serial [6] or parallel [7, 8] manipulators have been studied or a manipulator with a specific fault-tolerant property has been designed [6, 9]. Within the literature dealing with the control, the fault detection [10], fault isolation and identification [11] and fault recovery [5, 12, 13] are discussed. Various strategies such as model based or artificial intelligence (AI) controllers have been proposed for fault-tolerant manipulators.

Serial link manipulators (SLM) have received significant attention in the robotics community because of their wide range of applications. Fault-tolerant design of the SLMs can be achieved by adding extra kinematic redundancy [3]. By this redundancy, manipulators are considered as serial link redundant manipulators (SLRM). It has been shown that SLRMs can maintain their dependability to perform the required [14] or prioritized tasks [15] despite joint failures. It has also been observed that adding kinematic redundancy not only improves the fault tolerance specifications of the manipulators, but also promotes other static or dynamic properties including higher dexterous movements [16], lower maintenance time and repair costs, obstacle avoidance [17] and capability for motion planning and control with multiple constraints[15, 18, 19].

However, it should also be noted that having kinematic redundancy does not guarantee the fault-tolerant operation of redundant manipulators [6], because the kinematic redundancy has to be efficiently used to achieve the fault tolerance. In [20] the number of required joint redundancies is investigated by applying the joint failure probability and total reliability of the manipulator. In [6, 21] fundamental limits of optimal fault-tolerant configuration for SLRMs are studied and the constraints of the optimal fault tolerance are presented.

Identifying a fault-tolerant operation of a redundant manipulator in the forward or inverse kinematic domains is hardly possible due to the nonlinearity of the forward or inverse kinematic equations. Therefore, commonly the properties of the Jacobian matrix or null space of the Jacobian matrix are used by the researchers. For example, the jacobian matrix is used to define fault tolerance indices that can be deployed to perform fault tolerance analysis or design of fault-tolerant controllers. In [6, 21, 22], the main properties of Jacobian matrix or null space of the Jacobian matrix have been analyzed.

Using the kinematic redundancy, the robotic manipulators can be positioned in an optimal configuration for better performance in fault-tolerant operation. In these configurations, the fault tolerance will be more convenient than other configurations because there will be lower required reconfiguration. For example, if a surgical robot cuts a patient’s body with a surgical knife, then an appropriate configuration can minimize the undesirable behavior of the robot when locked-joint failures occur. This
will add to the safety of the patient and reliability of the robot.

In the optimal configurations for a fault-tolerant manipulator, the faulty manipulator can continue its task with a minimum relative reconfiguration. This suggests the ratio of the norm of the required compensating velocities to the norm of the joint velocities as an objective function that can be minimized. This problem is called optimal configuration with a minimum relative joint velocity jump (RJVJ). The answer to this problem is useful for fast fault recovery of the manipulators.

**Problem statement:** if a SLRM is capable of performing fault-tolerant motion, then what is the optimal configuration for the given pose that requires a minimum RJVJ for fault tolerance?

With regard to this problem, the fault-tolerant motion of the manipulators has been addressed in the literature with minimum joint velocity jump (JVJ) and minimum end effector (EEF) velocity jump. For example, the minimum JVJ for fault recovery from a joint failure has been obtained for a given configuration in [23, 24]. Using the minimum EEF velocity jump strategy, the control law for joint velocity reconfiguration with minimum EEF velocity jump and minimum JVJ has been obtained for a class of static non linear systems in [5] and for SLRMs in [12, 13]. It is clear to see that this literature deals with the post failure operation of the manipulator because they aim to tolerate the faults that can occur in a given configuration of the manipulator. In contrast, in this paper, the optimal configuration for fault recovery is investigated which is related to prior to the failure occurrence time.

The other contribution of this paper is to use the condition number for fault tolerance of manipulators. Note that the condition number is commonly used as an isotropic dexterity index of manipulators [25-28] while the relative manipulability and worst case dexterity are used mostly to obtain the optimal fault-tolerant configurations of the manipulators [6, 9, 29]. There is not a great deal of work on condition numbers for fault tolerance analysis of the manipulator. For example, the condition number has been proposed for fault tolerance in [3, 30] but it has not been deployed. This work is an extension of the observation for the fault tolerance property of the condition number in [31] and the derived control law for fault tolerance in [5, 12, 13]. In this paper, different aspects of the properties of the condition number for SLRMs are discussed. This leads to the introduction of the optimal configuration for fault recovery. Through a framework that is associated to the condition number, those configurations of the manipulators that can recover the fault with a minimum RJVJ are defined.

This paper is organized as follows. The Jacobian matrix of SLRM subjected to locked-joint failures is presented in section II. Then, the fault tolerance indices are reviewed in section III. Following that, the condition number and its application for fault tolerance are addressed in section IV where the optimal configurations for fault recovery of the manipulator are defined by using the optimality of the condition number. The condition number of a 4-DoF planar manipulator is studied from the dexterity and the fault-tolerant point of views in section V. Then, in section VI, it is shown that the optimal configuration is the solution of the singular value optimization problems. The optimal configurations for fault recovery are obtained for a case study in section VII. Next, the research presented in this paper is compared with other literature in section VIII. Finally, the conclusion remarks are presented in section IX.

## II. KINEMATICS OF FAULTY MANIPULATORS

The forward kinematics of a manipulator relates the joint angles to the EEF position/orientation by:

\[ x = f(q) \]

\[ q = [q_1, q_2, \ldots, q_n]^T \]

\[ x = [x_1, x_2, \ldots, x_m]^T \]

where joint variables (2) define the configuration space, position/orientation variables (3) define the workspace of the manipulator, \( n \) is the dimension of the configuration space and \( m \) is the dimension of the workspace.

The inverse kinematics is given by:

\[ q = f^{-1}(x) \]

The manipulator with \( n \)-DoF (Degrees of Freedom) is redundant \( n > m \), and the degrees of kinematic redundancy (DoR) is obtained by \( n - m \). It is assumed that the manipulator has only revolute joints.

The Jacobian matrix of the manipulators is obtained by:

\[ J = \left[ \frac{\partial f}{\partial q} \right] \in R^{m \times n} \]

where:

\[ \frac{\partial f}{\partial q} = J(q) \]

\[ J = [j_1, j_2, \ldots, j_{k-1}, j_k, j_{k+1}, \ldots, j_n] \]

If \( j_k \in R^m \) is the \( k \)-th column of the Jacobian matrix, then this column indicates the contribution of the \( k \)-th joint into the EEF translation and orientation velocity. The Jacobian matrix of a redundant manipulator with locked-joint failures can be obtained from the above Jacobian matrix, because if the manipulator’s \( k \)-th joint has failed, then this joint will not be able contribute into the EEF velocity [6, 21]. Therefore, the faulty manipulator Jacobian matrix can be obtained simply by replacing a 0 vector instead of the \( k \)-th column of the original Jacobian matrix that will be:

\[ [j_1, j_2, \ldots, j_{k-1}, 0, j_{k+1}, \ldots, j_n] \]
Then the velocity equation for the faulty manipulator subject to the k-th joint failure is obtained by
\[ \dot{x} = kJ q_k \]
where
\[ kJ = [j_1 \ j_2 \ \cdots \ j_{k-1} \ j_{k+1} \ \cdots \ j_n] \]
\[ kq = [\dot{q}_1 \ \dot{q}_2 \ \cdots \ \dot{q}_{k-1} \ \dot{q}_{k+1} \ \cdots \ \dot{q}_n] \]
The Jacobian matrix (10) is called the k-th reduced Jacobian matrix. There are n reduced Jacobian matrices associated to the failure of joint 1, 2, ..., n that are shown by
\[ ^1J, ^2J, \ldots, ^nJ \]
With a similar column elimination approach, if the manipulator has f faults, then the reduced Jacobian matrices are obtained with the permutation of f zero vectors replaced into the original Jacobian matrix showing the failures of the corresponding joints and then eliminate the columns. For example for the case of two locked-joint failures, there are \( \frac{n(n-1)}{2} \) reduced matrices that are shown by \(^k_J\) where \( k, i = 1, \ldots, n \) and \( i > k \) represents the faulty joints. This reduced matrix is obtained by eliminating the zero columns of the following matrix
\[ [j_1 \ \cdots \ j_{k-1} \ 0 \ j_{k+1} \ \cdots \ j_{i-1} \ 0 \ j_{i+1} \ \cdots \ j_n] \]
The velocity equation of the manipulator when the k-th and i-th joints are failed is then obtained by
\[ \dot{x} = ^{k,i}J ^{k,i} q \]
where
\[ ^{k,i}J = [j_1 \ \cdots \ j_{k-1} \ j_{k+1} \ \cdots \ j_{i-1} \ j_{i+1} \ \cdots \ j_n] \]
\[ ^{k,i}q = [\dot{q}_1 \ \cdots \ \dot{q}_{k-1} \ \dot{q}_{k+1} \ \cdots \ \dot{q}_{i-1} \ \dot{q}_{i+1} \ \cdots \ \dot{q}_n] \]

### III. FAULT TOLERANCE INDICES

The most common local fault tolerance indices of manipulators are shown in Table 1. These indices are based on properties of the Jacobian matrix. The indices such as manipulability (17), (20), relative manipulability (18) and worst case dexterity (21) have been studied for fault tolerance of the manipulator by different of researchers. For instance, the optimal configuration and optimal Jacobian matrix for fault-tolerant manipulator have been addressed based on relative manipulability in [6, 21, 22, 32, 33] and based on worst case dexterity in [9, 16, 34].

Condition number (19) has been proposed for the isotropic dexterity of the manipulators in [25-28]. It has also been proposed for the fault tolerance in [3, 16, 30], but even in this literature, the authors have selected the worst case dexterity for the analysis of the fault tolerance. There is lack of an extensive study of condition number for fault tolerance of the redundant manipulators.

<table>
<thead>
<tr>
<th>Table 1 - Common Local Fault-Tolerant Indices</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Index name</strong></td>
</tr>
<tr>
<td>Manipulability</td>
</tr>
<tr>
<td>Relative Manipulability</td>
</tr>
<tr>
<td>Condition Number</td>
</tr>
<tr>
<td>Dynamic Manipulability</td>
</tr>
<tr>
<td>Worst Case Dexterity</td>
</tr>
<tr>
<td>Null space matrix</td>
</tr>
</tbody>
</table>

### IV. CONDITION NUMBER FOR FAULT TOLERANCE

#### A. Singular value decomposition and norms

If \( J \in \mathbb{R}^{m\times n} \) is a Jacobian matrix at a given EEF pose, then the singular value decomposition (SVD) of this Jacobian matrix is
\[ J = U \Sigma V^T \]
where \( U \in \mathbb{R}^{m\times m} \) is an orthogonal matrix, \( \Sigma \in \mathbb{R}^{m\times n} \) is a diagonal matrix consisting the singular values of \( J \) and \( \Sigma = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_l, 0, ..., 0) \) where \( l \leq \text{min}(m, n) \).

Also \( V \in \mathbb{R}^{n\times n} \) is an orthogonal matrix and is provided by the Eigen vectors of \( J^TJ \) and the vectors of the orthogonal base of the null space of the Jacobian matrix. Note that for SLRMs \( m < n \) and hence the manipulators are not in singular configuration, therefore \( l = m \).

In this paper norm of matrices and vectors are often used. The \( L_2 \) norm of a velocity vector is defined by
\[ \|q\|_2 = \sqrt{q^T q} = \sqrt{\sum_{j=1}^{n} q_j^2} \]
and \( L_2 \) norm of a matrix is defined by
\[ \| \mathbf{J} \| = \max_q \left\| \mathbf{J} \mathbf{q} \right\| \mathbf{q} \in \mathbb{R}^n, \| \mathbf{q} \| = 1 \]  \quad (25) \\

B. Modeling a fault using a matrix perturbation methodology

Generally, in the existing literature, the modeling of the locked-joint failure is performed by using the method that was introduced in section II, for example in equations (9)-(11) [6, 21]. But a fault can be also modeled by using a perturbation method. In this method, any locked joint failure perturb to the velocity equation because it affects both the faulty joint velocity and the Jacobian matrix [12, 13, 31]. The perturbed velocity equation is given by

\[ \left( \mathbf{J} + \Delta \mathbf{J} \right) \left( \mathbf{q} + \Delta \mathbf{q} \right) = \mathbf{\dot{x}} + \Delta \mathbf{\dot{x}} \]  \quad (26)

where \( \Delta \) indicated the perturbation into the parameter due to the faults. Then by using (6) and rearranging (26), the change into the velocity of the EEF of the manipulator is obtained by

\[ \Delta \mathbf{\dot{x}} = \mathbf{J} \Delta \mathbf{q} + \Delta \mathbf{J} (\mathbf{q} + \Delta \mathbf{q}) \]  \quad (27)

If the manipulator is fault-tolerant, then EEF velocity is required to remain with no change. Therefore it is essential to have \( \Delta \mathbf{\dot{x}} = 0 \) and as the result

\[ \mathbf{J} \Delta \mathbf{q} = -\Delta \mathbf{J} \mathbf{q} \]  \quad (28)

where \( \Delta \mathbf{q} \) is the change into the joint velocities that aims to tolerate the fault. The above equation can be solved to determine the required joint velocity jump by using pseudo inverse. The pseudo inverse depends to the column rank of the matrix [12, 13, 31].

The joint velocity jump to maintain the velocity of the EEF is determined by

\[ \Delta \mathbf{q} = -\mathbf{J}^* \Delta \mathbf{J} (\mathbf{q} + \Delta \mathbf{q}) \]  \quad (29)

where \( \mathbf{J}^* = (\mathbf{J}^T \mathbf{J})^{-1} \mathbf{J}^T \) when \( \mathbf{J} \) is a rank deficient and \( \mathbf{J}^* = \mathbf{J}^T (\mathbf{J} \mathbf{J}^T)^{-1} \) for full rank matrices [12, 13, 31].

C. Condition number

Using Appendix-C and applying the norm inequality to the equation (29) results to

\[ \| \Delta \mathbf{q} \| \leq \| \mathbf{J}^* \| \| \Delta \mathbf{J} \| \| \mathbf{q} + \Delta \mathbf{q} \| \]  \quad (30)

and consequently

\[ \frac{\| \Delta \mathbf{q} \|}{\| \mathbf{q} + \Delta \mathbf{q} \|} \leq \| \mathbf{J}^* \| \frac{\| \Delta \mathbf{J} \|}{\| \mathbf{J} \|} \]  \quad (31)

where \( \frac{\| \Delta \mathbf{q} \|}{\| \mathbf{q} + \Delta \mathbf{q} \|} \) is the ratio of the required change in \( \mathbf{q} \) to tolerate the fault. This shows \( \| \mathbf{J}^* \| \| \mathbf{J} \| \) as an upper bound for the required relative change of \( \mathbf{q} \). This upper bound only depends to the properties of the Jacobian matrix. It is also easy to see that

\[ \frac{\| \Delta \mathbf{q} \|}{\| \mathbf{q} + \Delta \mathbf{q} \|} \leq \| \mathbf{J}^* \| \frac{\| \Delta \mathbf{J} \|}{\| \mathbf{J} \|} \]  \quad (32)

Showing \( \frac{\| \Delta \mathbf{q} \|}{\| \mathbf{q} + \Delta \mathbf{q} \|} \frac{\| \Delta \mathbf{J} \|}{\| \mathbf{J} \|} \) is bounded by \( \| \mathbf{J}^* \| \| \mathbf{J} \| \).

D. Reconfiguration bounds and condition number

The definition of condition number [35] is given by

\[ \kappa(q) = \max \left\{ \frac{\| J_\mathbf{\tilde{q}} \|}{\| J_\mathbf{\tilde{q}} \|} : \| \mathbf{\tilde{q}} \| = \| \mathbf{\tilde{q}} \| = 1 \right\} \]  \quad (33)

where \( \| \mathbf{\tilde{q}} \|, \| \mathbf{\tilde{q}} \| \) are two unit norm joint velocity vectors that are related to the the input directions which result to the maximum and the minimum gain of the Jacobian matrix. It has been shown that \( \max(\| J_\mathbf{\tilde{q}} \|) = \sigma_{\max} \) and \( \min(\| J_\mathbf{\tilde{q}} \|) = \sigma_{\min} \) [35]. This results to the common definition of the condition number using the ratio of the the maximum to the minimum singular values as

\[ \kappa(q) = \frac{\sigma_{\max}}{\sigma_{\min}} \]  \quad (34)

It is also shown [35] than the condition number is equivalent to \( \| \mathbf{J}^* \| \| \mathbf{J} \| \), therefore \( \kappa(q) = \| \mathbf{J}^* \| \| \mathbf{J} \| \).

E. Fault tolerance and condition number

Singular values are used for dexterity analysis of the manipulator [18] because each singular value is the length of the corresponding Eigen vector in the image space of the Jacobian matrix. Therefore, the condition number is commonly considered as the isotropic dexterity index of the manipulators. By this concept, minimization of the condition number results in the maximization of the isotropic dexterity of the manipulator [25-28].

The present paper aims to investigate the condition number for the fault tolerance of manipulators. It is observed from (31) that condition number can be used for the fault tolerance. The connection between the fault tolerance and the condition number is achieved by using the perturbation
method that was presented in section IV.C. Based on this observation, if the condition number is small, then the Jacobian matrix is called a well-conditioned matrix. Therefore, it is logical to name the configuration with well-conditioned configuration. In these configurations through a little change into the joint velocities, the faulty manipulator EEF velocity can be maintained as that of the healthy manipulator. In contrast, if the condition number is large which is for ill-conditioned Jacobian matrices, then the configuration is named ill-conditioned configuration. In ill-conditioned configurations, the fault requires a large change into the joint velocities to maintain the EEF velocity. In conclusion, the configurations with minimum condition number are optimal for fault tolerance, because the manipulator will be able to rapidly reconfigure itself to tolerate the locked-joint failures.

It has to be noted that the condition numbers (34) are valid within the other norm domains such as Frobenius or \( l_\infty \) norms. However, the physical interpretation of those condition numbers and their applicability for fault tolerance requires further research.

F. Properties of condition number

For a SLRM at a given EEF pose and a given joint velocities the following properties exist.

1. For all configurations \( \kappa(q) \geq 1 \).
2. \( \kappa(q) \) is a local index suitable for dexterity analysis and for fault tolerance analysis.
3. The lower value for \( \kappa(q) \) is the better isotropic dexterity of the manipulators.
4. The lower value for \( \kappa(q) \) provides a higher fault tolerance property for the configuration. This is because, if the manipulator is subjected to a locked joint failure, then it is possible to maintain the EEF velocity with a small relative reconfiguration.
5. The upper bound of the change into joint velocity is defined by \( \| \Delta q \| \leq \kappa(q) \frac{\| \Delta J \|}{\| J \|} \| \Delta J + u \| \) (35)
6. This upper bound is general and it is valid for any single and multiple joint failures.
7. The bound is valid for other definitions of norms including \( l_1, l_2, \ldots, l_n \) and Frobenius norms.

G. Condition number and single locked-joint failures

The perturbation model of equation (25) requires an input to compensate the effect of the failure. If the reconfiguration input is shown by \( u \) then

\[
(J + \Delta J)(\dot{q} + \Delta \dot{q} + u) = \dot{x} + \Delta \dot{x}
\]

where \( \Delta \dot{x} \) is the velocity jump at EEF, \( \Delta J \) is the Jacobian perturbation, \( \Delta \dot{q} \) is the change of joint velocities, and \( \Delta \dot{q} + u \) is the total joint velocity change for compensation of the failure.

For the case of the \( k \)-th joint failure, the perturbations are obtained by

\[
\Delta J = \begin{bmatrix} 0 & \ldots & -j_k & \ldots & 0 \end{bmatrix}
\]

\[
\Delta \dot{q} = \begin{bmatrix} 0 & \ldots & -\dot{q}_k & \ldots & 0 \end{bmatrix}
\]

and similar to the approach used to obtain equation (31), the ratio of the total joint velocity change is obtained by

\[
\frac{\| \Delta \dot{q} + u \|}{\| \dot{q} + \Delta \dot{q} + u \|} \leq \kappa(q) \frac{\| \dot{J} \|}{\| J \|}
\]

Therefore, the upper bound of the relative joint velocity jump for the case of single locked-joint failures is \( \kappa(q) \frac{\| \dot{J} \|}{\| J \|} \).

V. CASE STUDY I- CONDITION NUMBER FOR DEXTERITY AND FAULT TOLERANCE

A 4-DoF planar manipulator with the Denavit Hartenberg (DH) parameters presented in Table 2 is used for the case studies in this paper. The robot is modeled in Matlab Robotics Toolbox [36]. The manipulator in the configuration that is shown in Figure 1 has the parameters that are presented in Table 3.

\[
\begin{bmatrix} 0.334 & 0.588 \end{bmatrix} m
\]

Figure 1. The 4-DoF manipulator of the case study I and II

<table>
<thead>
<tr>
<th>Joint</th>
<th>( s_i ) (m)</th>
<th>( d_i ) (m)</th>
<th>( \alpha_i )</th>
<th>( \theta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
<td>0.45</td>
<td>0</td>
<td>( \theta_1 )</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>0.32</td>
<td>0</td>
<td>( \theta_2 )</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>0.18</td>
<td>0</td>
<td>( \theta_3 )</td>
</tr>
<tr>
<td>4</td>
<td>0.05</td>
<td>0.12</td>
<td>0</td>
<td>( \theta_4 )</td>
</tr>
</tbody>
</table>

TABLE 2- DH PARAMETERS OF A 4-DOF PLANAR MANIPULATOR USED IN THE CASE STUDY I AND II
### TABLE 3-CONFIGURATION AND PARAMETERS OF THE MANIPULATOR

<table>
<thead>
<tr>
<th>Joint</th>
<th>$q_{\text{deg}}$</th>
<th>$q_{\text{rad/s}}$</th>
<th>Parameter of configuration in Figure 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>0.05</td>
<td>$\sigma_{\text{min}} = 0.378$, $\sigma_{\text{max}} = 0.816$</td>
</tr>
<tr>
<td>2</td>
<td>70</td>
<td>0.40</td>
<td>$\kappa(q) = 2.165$</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>0.20</td>
<td>$x = [0.005, 0.002]' \text{ m/s}$</td>
</tr>
<tr>
<td>4</td>
<td>65</td>
<td>0.10</td>
<td>$x = [0.334, 0.558]' \text{ m}$</td>
</tr>
</tbody>
</table>

### A. Condition number and dexterity

In this case study, the dexterity and fault tolerance of the manipulator is studied at the EEF pose shown in Figure 1. In order to do this, we obtained 28000 different configurations where the manipulator remains in its current pose. Then, for these configurations we calculated the condition number from the corresponding Jacobian matrix. The condition number associated to the selected configurations is shown in Figure 2. From this figure, it is clearly observed that not all the configurations are well-conditioned, because they do not have a low value condition number. It also indicates that from the 28000 configurations, there is almost no configuration with a condition number lower than 1.895. Additionally, the worst ill-conditioned configuration has a condition number that is slightly more than 5.

Designing the manipulator with a low condition number is discussed by researchers in order to design isotropic manipulators. For example, such a design has been presented in [25, 26] that are providing a minimum condition number.

Two selected configurations with high and low condition numbers are indicated in Figure 3 and Figure 4. Physically, the high conditioning and low condition for these two configurations is meaningful, because intuitively, the motion of the EEF for the manipulator in Figure 3 is much more convenient than the motion of the manipulator EEF at the configuration shown in Figure 4. Additionally, if these configurations are considered for human arms and hand, then the configuration in Figure 3 is more dexterous or convenient than that in Figure 4 because, when we are doing daily tasks normally we choose the configurations in Figure 3, except if there is some sort of limitations or disability.
The present paper aims to show that the well-conditioned configurations are not only dexterous but also suitable for fault tolerant. The fault tolerance property of the well-conditioned configurations is explained in the following subsection.

### B. Condition number and fault tolerance

It was mentioned earlier that condition number has been proposed for fault tolerance in [3, 16], but it has not been deployed in the literature of fault tolerance. In the previous section, it was also indicated that the condition number can be utilized to define the well-conditioned configurations. In these configurations, the faults can be tolerated with a minimum relative joint velocity jump. Placing the manipulator in these configurations enhances the manipulator’s fault tolerance property because the manipulator can rapidly reconfigure itself in order to tolerate the fault.

From the framework in (39), if a failure occurs to the one of the manipulator joints, then it is expected that for well-conditioned configurations such as those in Figure 3 and Figures 5 (a), (b) and (c), the manipulator will require a low RJVJ to maintain the EEF velocity. However, for ill-conditioned configurations such as that in Figure 4, the manipulator will require high RJVJ to maintain the EEF velocity. This required RJVJ of the well-conditioned configuration of Figure 3 is nearly 60% lower from the required RJVJ for the ill-conditioned configuration of Figure 4.

To quantitatively show this hypothesis, the two configurations in Figure 3 and Figure 4 are used and it is assumed that the manipulator is in a fault-tolerant motion operation. The joint velocity of the manipulator is \( \mathbf{q} = \begin{bmatrix} 0.05 & 0.40 & 0.20 & 0.10 \end{bmatrix} \text{ rad / s} \). The EEF pose is \( \mathbf{x} = \begin{bmatrix} 0.334 & 0.588 \end{bmatrix}^T \text{ m} \) and the EEF velocity at the fault time is \( \dot{\mathbf{x}} = \begin{bmatrix} 0.005 & 0.002 \end{bmatrix}^T \text{ m / s} \).

If a locked joint failure occurs in the second joint of the manipulator. The JVJ to compensate this fault has infinite solution as the faulty manipulator is still redundant manipulator (a 3-DoF planar manipulator). The control law to maintain the EEF velocity with a minimum JVJ [5, 13] is used to obtain the JVJ.

In [5, 12, 13] it was shown that, the optimal reconfiguration command for fault recovery of the EEF velocity jump with the minimum joint velocity jump is obtained by

\[
\mathbf{u} = \mathbf{J}^\dagger \dot{\mathbf{j}} \mathbf{q},
\]

that is for single joint failures. Then \( \mathbf{u} \) is obtained from \( \mathbf{k} \mathbf{u} \) by inserting a zero in its \( \mathbf{k} \)-th row.

Using the above reconfiguration law, the JVJ is obtained for tolerating the fault, and the relative joint velocity jump is obtained from

\[
\frac{\|\Delta \mathbf{q} + \mathbf{u}\|}{\|\mathbf{q} + \Delta \mathbf{q} + \mathbf{u}\|},
\]

where \( \dot{\mathbf{q}} \) and \( \dot{\mathbf{q}} + \Delta \dot{\mathbf{q}} + \mathbf{u} \) are joint velocity vectors for the healthy and the faulty manipulator.
Using the reconfiguration law, the faulty manipulator’s EEF velocity will remain the same as that of the healthy manipulator. Table 4 provides the joint velocities prior to the failure and after the failure for the fault-tolerant motion.

**TABLE 4—JOINT VELOCITY JUMP FOR THE WELL-CONDITIONED CONFIGURATION OF FIGURE 3**

<table>
<thead>
<tr>
<th>Joint</th>
<th>( q^{(\deg)} )</th>
<th>( q^{(\text{rad/s})} )</th>
<th>( q^{(\text{rad/s})} )</th>
<th>( \text{JVJ}^{(\text{rad/s})} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>healthy joints</td>
<td>18.13</td>
<td>0.0050</td>
<td>0.0072</td>
<td>-0.0022</td>
</tr>
<tr>
<td>2nd joint failure</td>
<td>43.26</td>
<td>0.0046</td>
<td>0.0000</td>
<td>0.0046</td>
</tr>
<tr>
<td>3</td>
<td>81.54</td>
<td>0.0022</td>
<td>0.0062</td>
<td>-0.0040</td>
</tr>
<tr>
<td>4</td>
<td>9.04</td>
<td>0.0008</td>
<td>0.0024</td>
<td>-0.0016</td>
</tr>
</tbody>
</table>

A similar study has been performed to calculate the JVJ for the manipulator in the ill-conditioned configuration shown in Figure 4 and its result is indicated in Table 5.

**TABLE 5—JOINT VELOCITY JUMP FOR THE ILL-CONDITIONED CONFIGURATION OF FIGURE 4**

<table>
<thead>
<tr>
<th>Joint</th>
<th>( q^{(\deg)} )</th>
<th>( q^{(\text{rad/s})} )</th>
<th>( q^{(\text{rad/s})} )</th>
<th>( \text{JVJ}^{(\text{rad/s})} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>healthy joints</td>
<td>51.32</td>
<td>0.0066</td>
<td>0.0078</td>
<td>-0.0012</td>
</tr>
<tr>
<td>2nd joint failure</td>
<td>12.87</td>
<td>-0.0034</td>
<td>0.0000</td>
<td>-0.0034</td>
</tr>
<tr>
<td>3</td>
<td>103.11</td>
<td>0.0000</td>
<td>0.0007</td>
<td>-0.0007</td>
</tr>
<tr>
<td>4</td>
<td>146.12</td>
<td>0.0019</td>
<td>0.0038</td>
<td>-0.0019</td>
</tr>
</tbody>
</table>

By comparing the relative reconfiguration for joint velocity jumps for the both well and ill configurations that are shown in Tables 4 and 5, it is observed that the well-conditioned configuration requires 55.26% lower RJVJ.

Having a lower RJVJ for well-conditioned configurations is valid in other EEF poses and different configurations. This confirms the proposed hypothesis of this paper that the well-conditioned configurations are also suitable for fault tolerance.

This conclusion is used in the next section and the optimal configurations for fault tolerance of the manipulators are obtained from the minimum condition numbers.

**VI. OPTIMAL FAULT-TOLERANT CONFIGURATIONS**

From section IV.G, it is clearly seen that the minimization of the condition number can minimize the bound of the reconfiguration for tolerating the locked joint failures. Therefore, in the fault-tolerant operations, it is useful to find the configurations with low condition numbers as they require less reconfiguration. This section aims to show that by the minimization of the condition number of the Jacobian matrix, the optimal configuration for fault tolerance can be obtained.

To optimize the condition number, any of the following cost functions can be used

\[
I_f = \| J^T J \| \quad (40)
\]

\[
I_f = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \quad (41)
\]

The cost function in (40) is applicable if the unit inconsistency exists or the Jacobian matrix is non-homogen. This can be the case when both positional and orientational velocities are considered.

In this paper, we use (41) because only translational velocities are considered and the Jacobian matrix is a homogen. Then the minimization of (41) can be rewritten into three standard singular value (Eigen value) optimization problems [39].

**A. Singular value optimization-first method**

The first method for the optimizing of the equation (41) is to constrain the maximum singular value of the Jacobian matrix and then minimize the maximum Eigen value. This case results in the following constrained singular value maximization problem

\[
\begin{align*}
\max (\sigma_{\text{min}}) \\
\sigma_{\text{max}} \leq z_h
\end{align*}
\]

where \( z_h \) is a constant number.

The problem can be converted to

\[
\begin{align*}
\max (z) \\
\sigma_{\text{min}} > z \\
\sigma_{\text{max}} \leq z_h
\end{align*}
\]

Considering that for all singular values \( \sigma_i \leq \sigma_{\text{max}} \) when \( i=1,2,\ldots,m \) and \( \sigma_{\text{min}} > z \) then simply it is obtained \( \sigma_i - z > 0 \). In addition, by using the Sylvester theorem introduced in Appendix-B, the constraints of (43) can be written in terms of determinants of the principal sub-matrices of \( J^T J - z I \) where \( I \) is the identity matrix. Therefore, the maximization of the minimum singular value can be obtained by the following standard constrained optimization problem.

\[
\begin{align*}
\max (z) \\
\det(J^T J - z I) \geq 0 \quad i=1,2,\ldots,m \\
\sigma_{\text{max}} \leq z_h
\end{align*}
\]

**B. Singular value optimization-second method**

The second method is to constrain the lower bound of minimum singular value and then minimize the maximum singular value of the Jacobian matrix. Therefore

\[
\begin{align*}
\min (\sigma_{\text{max}}) \\
\sigma_{\text{min}} \geq z_i
\end{align*}
\]

Similar to the first method and by using the Sylvester condition, the minimization of the maximum singular value
can be obtained by the following standard constrained optimization problem.

\[
\begin{aligned}
\min_{q, z} & \quad \sigma_{\min} \\
\text{subject to} & \quad \det(-J^T J + zI_i) \geq 0, \quad i = 1, 2, \ldots, m \\
& \quad \sigma_{\min} \geq z_i
\end{aligned}
\] (46)

C. Singular value optimization-third method

This method converts the optimality problem into a multi-objective Eigen value optimization problem. Note that the Jacobian matrix is a positive definite matrix, therefore all the Eigen values of the Jacobian matrix and reduced Jacobian matrix are positive. Hence the minimization of \(-\sigma_{\min}\) results into the maximization of \(\sigma_{\min}\) therefore the multi-objective are defined by

\[I_f = \min_{q} (\sigma_{\max} - \sigma_{\min})\] (47)

Then using the Sylvester theorem, a multi-objective multi-constrained optimization problem is obtained by

\[
\begin{aligned}
\max_{q, \varepsilon} & \quad z_i \\
\text{subject to} & \quad \det(J^T J - zI_i) \geq 0, \quad i = 1, 2, \ldots, m \\
& \quad \det(\varepsilon I - J^T J) \geq 0, \quad i = 1, 2, \ldots, m
\end{aligned}
\] (48)

Note that in this method there are \(2m\) non equality constraints and two objective functions.

D. Selection between the optimization methods

From the concept of the dexterity, the second and third methods are preferred, because in the first optimization method, it is possible to converge to a configuration with a very low condition number and therefore a low total dexterity. In contrast, in the second and the third methods, the total dexterity is already preserved through the threshold or maximization of the minimum singular value. Between the last two methods, we selected the third optimization method and implemented the method in Matlab using the functions of optimization toolbox.

VII. CASE STUDY II– OPTIMAL FAULT-TOLERANT CONFIGURATIONS FOR THE 4-DoF MANIPULATOR

The 4-DoF manipulator with the DH parameters of Table 2 and with the positional parameters in Table 3 was used in this case study. Figure 6 indicates the singular values corresponding to the 28000 configurations of the case study I. The maximum and minimum singular values are used as the initial guess for the maximum and minimum singular values.

A. Optimization

The optimal configuration was obtained through the optimization process of (48) and via a multi-objective multi-constraints optimization method. Five results of the optimum configurations are indicated in Table 6. These results include the joint angles and the corresponding condition number for the five obtained optimal configurations. In this table, the joint angles are indicated in degree and the condition numbers are indicated in last row of the table.

![Figure 6. Maximum and minimum singular values for different configuration used for the initial guess for the optimizations problem](image)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Conf. No.1</th>
<th>Conf. No.2</th>
<th>Conf. No.3</th>
<th>Conf. No.4</th>
<th>Conf. No.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joint 1</td>
<td>20.1</td>
<td>24.2</td>
<td>25.8</td>
<td>22.5</td>
<td>19.9</td>
</tr>
<tr>
<td>Joint 2</td>
<td>38.8</td>
<td>29.6</td>
<td>26.4</td>
<td>33.6</td>
<td>54.5</td>
</tr>
<tr>
<td>Joint 3</td>
<td>77.7</td>
<td>94.1</td>
<td>98.3</td>
<td>92.4</td>
<td>37.6</td>
</tr>
<tr>
<td>Joint 4</td>
<td>26.3</td>
<td>7.8</td>
<td>7.9</td>
<td>4.5</td>
<td>90.2</td>
</tr>
<tr>
<td>(\kappa(q))</td>
<td>1.907</td>
<td>1.90</td>
<td>1.895</td>
<td>1.897</td>
<td>1.895</td>
</tr>
</tbody>
</table>

B. Discussion

From figure 2, it is understood that the optimal configuration is not a unique configuration as there is a whole bunch of well-conditioned configurations with similar level of condition number. The non uniqueness of the optimal configuration is simply observed from the results of Table 6 because the optimization does not converge to a specific solution. Actually, there are infinite numbers of well-conditioned configurations and the result of the optimization problem is very sensitive to the initial guess. However, in all the configurations in the Table 6, the values of the condition numbers are consistent.

C. Limitation and unit inconsistency

The dexterity indices are not well physically interpreted when they applied for the non homogenous Jacobian matrices. This is also correct for the condition number. This problem is general in most of the literature of the manipulators when different units are combined in the Jacobian matrix.

In this case we proposed using (40) as the cost function instead of the (41), but we haven’t deployed it in this paper. Such a deployment requires further research.

Furthermore, the generalization of the proposed method for manipulators with combined revolute and prismatic joints or the manipulators with positional and rotational Jacobian...
matrix require also further research. But the heuristic of the dexterous configurations for the fault tolerance is physically valid even for those manipulators.

Another limitation is that the proposed method is suitable only for locked-joint failures. For free swing joint failures and joint sensor failures, it is possible to consider a mechanical brake to lock the joints. In this case, the proposed method of this paper will be applicable for other types of joint failures.

VIII. COMPARISON OF THIS WORK WITH OTHER WORKS ON FAULT-TOLERANT MOTIONS

From the literature survey we found that condition number has been proposed for fault tolerance but has not been deployed to limit the bound of the RJVJ for fault recovery. This motivated the research that was presented in this paper aimed to promote the tolerating of the faults in the SLRMs using condition number. Then by using matrix perturbation methodology, a framework for finding the optimal fault-tolerant configurations is introduced. The main property of these configurations is that, they require a lower RJVJ to tolerate the locked-joint failures. In comparison to the work of the other researchers, the joints velocity jump and the EEF velocity jump have been addressed in [23, 24] and [12, 13, 31] to provide the control law for joint velocity reconfiguration. The reconfiguration laws result to minimum RJVJ for fault recovery. The other works in [6, 9, 16, 21, 22, 29, 30, 40-42] in the literature, are related to design of the optimal fault tolerant redundant manipulators and in them the dexterity or relative manipulability indices are used. These works are also different from the preset paper because the paper uses condition number as the index of fault tolerance.

IX. CONCLUSION

Matrix perturbation method was applied for modeling the locked-joint failure for serial link redundant manipulators. Then the condition number of the Jacobian matrix was studied and it was observed that in addition to the common use of the condition number for the isotropic dexterity, it can be useful to define the optimal configurations for fault tolerance. This fact led to introducing well-conditioned configurations from the optimality of the condition number. In these configurations, reconfiguring the manipulator to tolerate the failures requires a low relative reconfiguration. This property was demonstrated for a 4-DoF planar manipulator. The results showed that the reconfigurations required for fault tolerance in the well-conditioned configuration is much lower than that in the ill-conditioned configurations.

APPENDICES

Appendix-A

If a matrix is a positive definite then all the Eigen values are greater than zero. Therefore, if we suppose the singular values as \( \sigma_i \geq z > 0 \), \( i=1,2,...,n \) then any matrix by the singular values of \( \sigma_i - z \sigma_i - z \geq 0 \), \( i=1,2,...,n \) is a positive definite matrix.

The square of the singular values of the Jacobian matrix is the Eigen values of \( J^T J \) and \( \sigma_i^2 - z^2 \) are Eigen values of \( J^T J - z I \) and all the \( \sigma_i^2 - z^2 > 0 \) for \( i=1,2,...,n \) then \( J^T J - z I \) is a positive define matrix.

Appendix-B

Sylvester condition: A matrix is a positive semi-definite matrix if and only if all the determinants of the principal sub-matrices which are called principal minors are positive [35, 43].

Appendix-C

If A, B and C are three arbitrary matrices satisfying \( C = AB \) then following inequality is valid for all the norm definitions including \( l_1, l_2, ..., l_\infty \) and Frobenius norms.

\[ \|C\| \leq \|A\|\|B\| \]

REFERENCES


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