A Family of Planar Parallel Manipulators

Mats Isaksson

Abstract

A family of planar parallel manipulators is investigated and some novel members are proposed. The common feature of the studied manipulators is that the rotation axes of the actuated arms coincide. This feature makes it possible to rotate the whole arm system an infinite number of revolutions around the center of the manipulator. The result is a large workspace in relation to the footprint. Both 2- and 3-DOF variants are presented and the suitability of this family of manipulators for kinematic analysis is demonstrated. Thus, different methods to find optimal manipulability with respect to platform positioning and rotation have been analyzed.

I. INTRODUCTION

Parallel mechanisms for planar manipulation are increasingly used in high precision applications. The book [1] provides a comprehensive treatment of different architectures for planar manipulation based on three kinematic chains connecting the frame and the end-effector. One of the most commonly studied planar architectures in the literature is the 3–RRR manipulator. This paper investigates a specific family of parallel manipulators for planar manipulation based on RRR kinematic chains where the actuated arms have a common axis of rotation. The main advantage of using a common axis of rotation is that the achievable planar workspace is significantly increased.

In addition to being of interest in their own right the parallel manipulators studied in this paper are useful for analyzing and visualizing the effects of various optimization algorithms for structural parameters. Due to rotational symmetry, it is sufficient to study the manipulator properties in only one radial direction, significantly simplifying such analysis. One important design criteria for manipulators of the investigated family is the ratio between the lengths of the lower arm links and the upper arms. Here analytical Singular Value Decomposition (SVD) of the Jacobian is utilized to improve the understanding of this design criteria.

When designing parallel manipulators for robotic applications it is important to obtain movements that are useful for the robot tasks. One example of this is the design of the SCARA-Tau robot [2]. In that case it is important to obtain minimum rotation of the manipulated platform when it is positioned in the radial direction. This design task can easily be studied using members of the investigated manipulator family and a better solution than utilizing a triangular link arrangement will be presented.

Parallel manipulators usually have a limited range of achievable platform rotation compared to serial manipulators. One way to expand this range is by introducing actuator redundancy. An example of this is the actuation-redundant PKM Archi [3]. Utilizing one redundant kinematic chain allows members of the investigated manipulator family to achieve an infinite range of platform rotation in addition to the large positional workspace.

Although the mechanisms discussed here are for planar manipulation, many of the results found when analyzing these manipulators are immediately useful for similar spatial PKMs, in particular for the SCARA-Tau PKM [4].

II. DESCRIPTION OF THE INVESTIGATED MANIPULATORS

A. Manipulators with coinciding axes of rotation

All of the mechanisms for planar manipulation studied in this paper include two or more actuated upper arms, which can rotate indefinitely around a cylindrical base column as shown in Fig. 1. The actuators for the rotating upper arms are placed on the base column. Each upper arm is connected by at least one lower arm link to a manipulated platform. The lower arm links are passive in most cases but are also allowed to contain telescopic actuators. The upper arms, the lower arm links and the different horizontal sections of the platform can be arranged to operate in different planes. In this way collisions between upper arms and lower arm links or between lower arm links and platform are avoided in most cases. For all manipulators in Fig. 1, with the exception of the Triangular Duorot in Fig. 1(b), the lower arm links are not susceptible to horizontal bending. The possibility of using identical drivelines, identical upper arms and identical lower arm links means that the number of different components for a robot can be kept low, which is an advantage for the manufacture of the manipulators. Since the upper arms can rotate indefinitely around the base column, the workspace is large and this feature makes it possible to always use the closest trajectory between two ordered positions. The positional workspace of a manipulator of this type forms a disc with a hole in the center.

A few spatial PKMs that allow continuous rotation of the whole arm system around a central column are proposed in the literature. One of the earliest variants of this type is found in [5]. For this manipulator the axis of rotation of the uppermost actuated arm is angled 90 degrees relative to the axis of rotation of the other two actuated arms. One drawback of this manipulator is that the link connecting the uppermost actuated arm and the manipulated platform is susceptible to torsion.

M. Isaksson is with the Centre for Intelligent Systems Research (CISR), Deakin University, Australia. E-mail: mei@deakin.edu.au
Fig. 1. Members of the investigated family of planar parallel manipulators. Each manipulator consists of a cylindrical base column (grey), upper arms (green), lower arm links (blue), a platform structure (black) and rotating joints (red). In (h) there is also a planar prismatic joint that decouples the upper (grey) and lower (black) sections of the platform structure.

Spatial PKMs utilizing a common axis of rotation are proposed in [4] and [6]. The manipulator proposed in [6] and some of the variants proposed in [4] employ a crankshaft mechanism to achieve rotation of the manipulated platform. The manipulator described in [6] has the advantage that it can achieve continuous platform rotation without the use of redundant actuators. The manipulator in Fig. 1(g) is a planar variant of the mechanisms proposed in [4] and [6]. The author has not previously encountered any manipulator of the type proposed in Fig. 1(h), where an over-constrained manipulated platform is avoided by introducing a passive telescopic link in the platform structure.

B. Duorot manipulators

A Duorot manipulator has two actuated rotating upper arms. Examples of Duorot manipulators are shown in the first four illustrations in Fig. 1. Each of the two upper arms is connected to a platform by one or two lower arm links. The most basic manipulator of this type, using one lower arm link per upper arm, is shown in Fig. 1(a). Each of the two lower arm links only impose one constraint on the three planar DOFs and hence the manipulator can only be used to control a 2-DOF position and not a platform rotation.

It is possible to achieve a two-link Duorot manipulator with a predictable platform rotation. The simplest approach is to remove one of the two joints on the platform structure (black) in Fig. 1(a). Another example is shown in Fig. 1(b). One joint has been removed and one of the lower arm links has been expanded to include a platform. The rotating joint between the platform structure (black) and the upper arm imposes two constraints on the planar DOFs of the platform and the other lower arm link imposes one constraint, so all three planar DOFs of the platform are constrained when the actuators are locked. Since only two actuators are used, only two DOFs can be controlled and when the planar position of the platform is manipulated, its platform rotation is dependent on this position. The drawback of a manipulator of this type compared to all other manipulators in Fig. 1 is that horizontal bending can occur in the link containing the platform. Another interesting 2-DOF variant can be achieved based on the structure in Fig. 1(a). By adding a passive $RP$ kinematic chain between the base column and the manipulated platform the platform orientation is kept constant for positions in the same radial direction. The additional kinematic chain does not place any extra constraints on the platform position. The drawback of this solution is that the added kinematic chain is susceptible to horizontal bending and increases the inertia of the manipulator. Similar variants with reduced inertia can be achieved by removing one of the two actuated $RRR$ kinematic chains and instead actuating one of the two joints in the $RP$ kinematic chain.

To achieve a manipulator with predictable platform orientation where the links are not susceptible to horizontal bending, an additional lower arm link must be included in the mechanism. This link could be attached in different ways. Two possibilities are shown in Fig. 1(c) and Fig. 1(d). For both of these manipulators all three planar DOFs are constrained when the actuators are locked, but since only two actuators are used to actuate three DOFs, it is not possible to actuate the x-position, y-position and platform rotation independently. Two out of three planar DOFs can be fully controlled while the third will vary dependent on the other two. For the manipulator in Fig. 1(d) the platform rotation is dependent only on the angle of the upper arm where the parallelogram is attached. An improvement of this design is given by the manipulator shown in Fig. 1(c). Using a triangular configuration of two of the lower arm links is an idea introduced in [7] and analyzed in [2]. The advantage of this arrangement is that the dependence between platform position and orientation is significantly decreased.
C. Triorot and Quatrorot manipulators

A Triorot manipulator for planar manipulation has three actuated rotating upper arms. Each upper arm is connected to a platform by a lower arm link. A general form of this manipulator is shown in Fig. 1(e). A manipulator of this type has three controllable planar DOFs. The Decoupled Triorot shown in Fig. 1(f) is an interesting special case of the general manipulator, where two of the platform joint positions are chosen to have the same planar position but different height. Using this arrangement the platform position in the plane is controlled by two kinematic chains and the third kinematic chain can only control the platform rotation, hence the name Decoupled Triorot. For this manipulator it is apparent that two Type 2 singularities [8] per 360 degree rotation of the platform exist. Analysis of the Jacobian determinant shows that this is also the case for the Basic Triorot in Fig. 1(e).

The workspace of a Triorot manipulator is large compared to most planar parallel manipulators but the range of platform rotation is still limited by the two Type 2 singularities occurring during platform rotation. By introducing actuation redundancy according to Fig. 1(g) these Type 2 singularities disappear and infinite platform rotation is possible in most positions. The manipulator in Fig. 1(g), where one kinematic chain consisting of an actuated upper arm and a passive lower arm link has been added, is named a Decoupled Quatrorot. The drawback of this manipulator is that its manipulated platform is over-constrained, meaning that translational stress can be introduced in the horizontal section of the platform crankshaft. This needs to be handled by the control loops, for example by letting one of the actuated upper arms be force controlled. An over-constrained platform can be avoided by replacing the horizontal section of the platform crankshaft with a passive telescopic link as shown in Fig. 1(h). Another feature of the telescopic link is that the length of this link can be modified by ordering a larger or smaller circle of the upper end of the crankshaft, thereby introducing a transmission with variable gear ratio for the control of the platform rotation.

III. Kinematics

A. Modeling

The naming conventions for the kinematic parameters of the investigated manipulators are shown in Fig. 2. For simplicity only parameters for a Duorot manipulator are included, but more upper arms (A₃, A₄), lower arm links (L₃₁, L₄₁) and joints (U₃₁, P₃₁, U₄₁, P₄₁) could be added using the same conventions, giving additional parameters (d₃₁, l₃₁, φ₃₁, d₄₁, l₄₁, r₄₁, φ₄₁). The manipulator in Fig. 1(b) is a special case and not covered by these conventions. Its kinematics is of the same type as the manipulator in Fig. 1(c).

A fixed coordinate system F is defined in the bottom center of the cylindrical base column with its x-axis pointing outwards and its z-axis pointing upwards. The upper arms are named Aᵢ, where i is 1 for the lowest upper arm. The joints on the upper arms are denoted Uᵢⱼ, where the index i refers to which upper arm the joint is attached to and the index j is used to differentiate between joints on the same upper arm. Each joint Uᵢⱼ is positioned a distance dᵢⱼ from the center of the base column. The joint U₁₁ does not necessarily have to be closest to the origin of F. The angles of the upper arms qᵢ are defined relative to the x-axis of F. The position of each joint on the upper arms is

\[
\mathbf{r}_{\text{ui}} = \begin{pmatrix} u_{ijx} \\ u_{ijy} \end{pmatrix} = \begin{pmatrix} d_{ij}\cos(q_i) \\ d_{ij}\sin(q_i) \end{pmatrix}.\]

The lower arm links Lᵢⱼ have the length lᵢⱼ. The indices i and j are the same as for the joints Uᵢⱼ. Each lower arm link Lᵢⱼ is attached to a joint Pᵢⱼ on the manipulated platform. A coordinate system T is attached to the platform, with its origin
in $P_{11}$. For the variant in Fig. 1(a) the orientation of $T$ is not relevant. For the variants in Fig. 1(c) - 1(h) the y-axis of $T$ is defined either by the direction between $P_{11}$ and $P_{21}$, or in the cases where $P_{11}$ and $P_{21}$ have the same planar position, by the direction between $P_{11}$ and $P_{31}$. The latter is the case for the variants in Fig. 1(f), 1(g) and 1(h). The positions of the joints $P_{ij}$ can be described in the coordinate system $T$ by using one distance $r_{ij}$ and one rotation $\phi_{ij}$, defined relative to the positive y-axis of $T$:

$$T_{P_{ij}} = \left(\begin{array}{c}
-r_{ij}\sin(\phi_{ij}) \\
-r_{ij}\cos(\phi_{ij})
\end{array}\right).$$

(2)

The values of $r_{11}$, $\phi_{11}$ and $\phi_{21}$ are by definition zero. For the manipulators in Fig. 1(f), 1(g) and 1(h) the values of $r_{21}$ and $\phi_{31}$ are also zero. For the last two manipulators $r_{41} = r_{31}$ while $\phi_{41}$ is zero. The position of $T$ in the fixed coordinate system $F$ is given by the translations $x$ and $y$ while the rotation of $T$ in $F$ is given by the yaw angle $\phi$. Using these conventions the coordinates of the platform joints in the fixed coordinate system $F$ are given by

$$F_{P_{ij}} = \left(\begin{array}{c}
 p_{ijx} \\
p_{ijy}
\end{array}\right) = \left(\begin{array}{c}
 x - r_{ij}\sin(\phi + \phi_{ij}) \\
y + r_{ij}\cos(\phi + \phi_{ij})
\end{array}\right).$$

(3)

B. Forward and Inverse Kinematics

The forward kinematics solutions for the Duorot and the redundant Quatrorot manipulators are uncomplicated. The forward kinematics solutions for the Triorot manipulators are basically the same as for 3−RRR manipulators without coinciding axes of rotation and can be found in [9].

A solution to the inverse kinematics is obtained from Fig. 3. This figure describes how to determine one upper arm angle, $q_i$, knowing the position of one platform joint $P_{ij}$ and the arm lengths $d_{ij}$ and $l_{ij}$. Each upper arm angle $q_i$ is a sum of two terms:

$$q_i = q_{im} \pm q_{id},$$

(4)

where

$$q_{im} = \arctan2(p_{ijy}, p_{ijx}),$$

$$q_{id} = \arccos\left(\frac{p_{ijx}^2 + p_{ijy}^2 + d_{ij}^2 - l_{ij}^2}{2d_{ij}\sqrt{p_{ijx}^2 + p_{ijy}^2}}\right).$$

(5)

The first term, $q_{im}$, is calculated using only $p_{ijx}$ and $p_{ijy}$, given by (3). The second term, $q_{id}$, is obtained from $p_{ijx}$, $p_{ijy}$ and the arm lengths using the theorem of cosines. There are always two solutions on both sides of $q_{im}$ except if an upper arm and a lower arm link are collinear, in which case $q_{id}$ becomes zero.
For the Basic Duorot in Fig. 1(a) the platform yaw angle, $\phi$, is not controlled and it is straightforward to decide $q_1$ and $q_2$ from $x$ and $y$. For the other Duorot variants in Fig. 1 actuation of $q_1$ and $q_2$ leads to movement in $x$, $y$ and $\phi$. If the planar position is controlled the yaw angle $\phi$ is dependent on this position and has to be determined from $x$ and $y$. Since $P_{11}$ is the origin of the coordinate system $T$ the angle $q_1$ is calculated independently of $\phi$. Then $x$, $y$ and $q_1$ can be used to determine $\phi$ and thereafter the position of $P_{21}$ can be calculated from (3) and $q_2$ from (4). For the Parallel Trilink Duorot $\phi$ is directly dependent on $q_1$ ($\phi = q_1 - \phi_{12} - \frac{\pi}{2}$) while for a general three-link Duorot an expression for the yaw angle is derived by using the length equation for the lower arm link $L_{12}$:

$$(u_{12x} - p_{12x})^2 + (u_{12y} - p_{12y})^2 - l_{12}^2 = 0.$$  \hspace{1cm} (6)

Inserting the values from (1) and (3) in (6) and simplifying gives

$$C_1 + C_2 \sin(\phi + \phi_{12}) + C_3 \cos(\phi + \phi_{12}) = 0,$$  \hspace{1cm} (7)

where

$$C_1 = x^2 + y^2 + d_{12}^2 + r_{12}^2 - l_{12}^2 - 2d_{12}(xcos(q_1) + ysin(q_1)),$$
$$C_2 = -2r_{12}x + 2d_{12}r_{12}\cos(q_1),$$
$$C_3 = 2r_{12}y - 2d_{12}r_{12}\sin(q_1).$$

Equation (7) has the two solutions:

$$\phi = -\phi_{12} - 2\arctan\left(\frac{C_2 \pm \sqrt{C_1^2 + C_2^2 + C_3^2}}{C_1 - C_3}\right).$$  \hspace{1cm} (9)

The inverse kinematics for the 3-DOF manipulators in Fig. 1(e), 1(f), 1(g) and 1(h) can be immediately determined using (4), since in this case both the platform position and the platform rotation is controlled and $\phi$ is known.

IV. Kinematic Analysis

A. Lengths of upper arms and lower arm links

As mentioned previously, the members of the studied family of parallel manipulators are well suited for the analysis of different criteria for kinematic design. To exemplify this the Basic Duorot manipulator in Fig. 1(a) is used to study the effect of different ratios between the lengths of the lower arm links and the upper arms. For all planar manipulators with a common axis of rotation, analysis of the properties of the manipulators along a radial line starting from the center of the manipulator is sufficient. Due to symmetry these positions include all manipulator properties. An instinctive choice is to

The length equation for each of the lower arm links $L_{11}$ and $L_{21}$ is the same as in (6), except for using the indices 11 or 21 instead of 12. Differentiating those two equations with respect to time gives

$$J_x\dot{x} = J_q\dot{q}.$$  \hspace{1cm} (10)

For $y = 0$ the Jacobians are simplified to

$$J_x = \begin{pmatrix} -2(d_{11}\cos(q_1) - x) & -2d_{11}\sin(q_1) \\ -2(d_{11}\cos(q_1) - x) & 2d_{11}\sin(q_1) \end{pmatrix},$$  \hspace{1cm} (11)
$$J_q = \begin{pmatrix} -2d_{11}x\sin(q_1) & 0 \\ 0 & 2d_{11}x\sin(q_1) \end{pmatrix},$$  \hspace{1cm} (12)
$$J = J_x^{-1}J_q = \frac{1}{2} \begin{pmatrix} d_{11}x\sin(q_1) & -d_{11}x\sin(q_1) \\ d_{11}\cos(q_1) & d_{11}\cos(q_1) \end{pmatrix}.$$  \hspace{1cm} (13)

The corresponding determinants are
The Basic Duorot has Type 2 singularities where the lower arm links are collinear. The exception is configurations where all four links (upper arms and lower arm links) are collinear which are Type 3 singularities [8]. If two identical kinematic chains are used there is one possible Type 2 singularity for \( y = 0 \). This singularity occurs when the lower arm links are parallel to the y-axis and can be avoided if \( l_{11} > d_{11} \). The Type 3 singularities for \( y = 0 \) occur in all solutions to \( \sin(q_1) = 0 \).

Performing an SVD of the Jacobian matrix \( J = USV^T \) in any position where \( x > 0, y = 0 \) leads to a matrix \( V \) consisting of two output singular vectors, \( \mathbf{v}_i \) and a diagonal matrix \( \Sigma \), where the diagonal elements are the two singular values \( \sigma_i \). The directions of the two output singular vectors, \( \mathbf{v}_i \), in joint space are coupled to the directions of the two input singular vectors, \( \mathbf{u}_i \), in Cartesian space. The coupling values are given by the two singular values. For the Basic Duorot the vectors \( \mathbf{v}_i \) and \( \mathbf{u}_i \) are in all studied positions \( \mathbf{v}_1 = \mathbf{v}_x = 1/\sqrt{2}[1 \ 0]^T \) corresponding to \( \mathbf{u}_1 = \mathbf{u}_x = [1 \ 0]^T \) and \( \mathbf{v}_2 = \mathbf{v}_y = 1/\sqrt{2}[1 \ 1]^T \) corresponding to \( \mathbf{u}_2 = \mathbf{u}_y = [0 \ 1]^T \). The singular value corresponding to the amplification in the x direction is denoted \( \sigma_x \) and in the y-direction \( \sigma_y \). Depending on configuration and radial position sometimes \( \sigma_x \) is larger than \( \sigma_y \), sometimes they are equal and sometimes \( \sigma_y \) is larger than \( \sigma_x \). Utilizing the definition of SVD and the values of \( \mathbf{v}_i \) and \( \mathbf{u}_i \) from above the Jacobian can be written

\[
J = USV^T = \sum_{i=1}^{2} \sigma_i \mathbf{u}_i \mathbf{v}_i^T = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_x & -\sigma_x \\ \sigma_y & \sigma_y \end{pmatrix}.
\]

Comparing the elements in (13) and (17) gives analytical expressions for the singular values:

\[
\sigma_x = \frac{d_{11} x \sin(q_1)}{\sqrt{2}(d_{11} \cos(q_1) - x)},
\]

\[
\sigma_y = \frac{x}{\sqrt{2}}.
\]

Using the theorem of cosines in the triangle formed by the base column, \( U_{11} \) and \( P_{11} \) to eliminate \( q_1 \) from (18) and factorizing the result gives

\[
\sigma_x = x \sqrt{-\frac{1}{d_{11}^2}(\frac{x}{d_{11}}+1+\frac{d_{11}}{d_{11}})(\frac{x}{d_{11}}+1+\frac{d_{11}}{d_{11}})(\frac{x}{d_{11}}+1+\frac{d_{11}}{d_{11}})} \sqrt{2((\frac{x}{d_{11}})^2+(\frac{d_{11}}{d_{11}})^2-1)}.
\]

An expression for the condition number \( (\kappa) \) is given by

\[
\kappa = \frac{\max(\sigma_x, \sigma_y)}{\min(\sigma_x, \sigma_y)}.
\]

The plots in Fig. 4 show normalized singular values \( (\sigma_x/d_{11} \text{ and } \sigma_y/d_{11}) \), ratios between singular values, condition number and normalized Jacobian determinant \( (\det(J)/d_{11}^2) \) as functions of normalized x-position \( (x/d_{11}) \). The plots are made for four different length ratios \( (l_{11}/d_{11}) \) between the lower arm links and the upper arms. The positions close to \( x/d_{11} = l_{11}/d_{11} - 1 \) are not reachable due to collision between the base column and the lower arm links. The exact size of the unreachable section depends on the radii of the cylindrical base column and the lower arm links.

According to (19) the value of \( \sigma_y \) and hence the speed amplification in the y-direction is independent of the arm length ratio. Regardless of the chosen arm lengths the two end positions of the workspace, \( x/d_{11} = l_{11}/d_{11} - 1 \) and \( x/d_{11} = l_{11}/d_{11} + 1 \), are both Type 3 singularities. When \( x \) approaches any of these positions the singular value \( \sigma_x \) and hence the determinant tend toward zero while the condition number tends toward infinity.

Depending on the ratio between the lengths of the lower arm links and the upper arms \( (l_{11}/d_{11}) \) different manipulator properties can be observed. In Fig. 4(a) this ratio is exactly one. In this case the two singular values \( \sigma_x \) and \( \sigma_y \) are equal in exactly one position, \( x/d_{11} = \sqrt{2} \), corresponding to \( q_1 = -45 \) degrees and an angle between \( A_1 \) and \( L_{11} \) (elbow angle) of 90 degrees. Since both singular values are equal the condition number is one and since both normalized singular values are one the normalized Jacobian determinant is also one. For the Basic Duorot this combination of arm lengths and position gives the largest equal speed amplification in both directions. For positions where \( x/d_{11} < \sqrt{2} \) the value of \( \sigma_x \) is larger than \( \sigma_y \).
meaning that the direction of maximum amplification is the x-direction. For positions where $x/d_{11} > \sqrt{2}$ the situation is the opposite and the amplification in the x-direction decreases until it becomes zero in the Type 3 singularity where $x/d_{11} = 2$. This choice of arm lengths leads to a relatively small section of the workspace with low condition number. If the position $x/d_{11} = 0$ was possible it represents a special case since an infinite number of solutions with different joint angles would be possible. The plotted solution corresponds to $q_1 = -180$ degrees as this represents a continuous solution compared to the solution for $x/d_{11} > 0$.

There is a fundamental difference between manipulators where $(l_{11}/d_{11})$ is smaller or larger than $\sqrt{2}$. For manipulators with ratio larger than $\sqrt{2}$ the singular value $\sigma_x$ corresponding to the gain in the y-direction is larger than the gain in the x-direction in all positions. This means that the condition number has a smooth derivative with a minimum value that is always larger than one. Figure 4(d) shows the system properties when the ratio is exactly $x/d_{11} = 1$. In this case $\sigma_x$ is always smaller than $\sigma_y$ except in exactly one position, $x/d_{11} = 1$, where they are equal. This position corresponds to $q_1 = -90$ degrees and elbow angles of 45 degrees.

The SCARA-Tau robot prototype [2] is a 3-DOF spatial PKM that also utilizes actuated upper arms with a common axis of rotation. This manipulator has an arm length ratio of 1.1/0.9 = 1.22. In Fig. 4(b) $l_{11}/d_{11} = 1.22$, which gives a condition number equal to one at the two positions where $q_1 = -60.0$ degrees and $q_1 = -120.0$ degrees.

A common approach to determine optimal structural parameters is to find parameters that maximize or minimize a criterion over the workspace. One such criterion, the global dexterity index (GDI) is defined in [10] as

$$GDI = \frac{\int_V \frac{1}{k(J)} dV}{V}. \quad (22)$$

Discretizing (22), using (19), (20), (21) and the fact that the manipulator has rotational symmetry leads to:

$$GDI \approx \frac{h_{x_n}}{\Delta x_n} \sum_{x_{n-1}}^{x_{n+1}} \min \left( 1, \frac{\sqrt{-(x_n+1)^2-r^2}/(x_n-1)^2-r^2}}{x_n^2+r^2-1} \right) \frac{1}{\max \left( 1, \frac{\sqrt{-(x_n+1)^2-r^2}/(x_n-1)^2-r^2}}{x_n^2+r^2-1} \right). \quad (23)$$

The variable $x_n = x/d_{11}$ is the normalized x-position, $h_{x_n}$ is the step size in $x_n$, $\Delta x_n = x_{n+1} - x_{n-1}$, and $r = l_{11}/d_{11}$ is the arm length ratio. The position $x_{n+1}$ equals $r+1$ while $x_{n-1}$ is close to $r-1$, but slightly larger to avoid collisions between the lower arm links and the base column. The exact value of $x_{n1}$ depends on the arm lengths and the radii of the base column and
the lower arm links. However, for all realistic choices of arm lengths and radii, the approximation $x_{n1} = r - 1$ influences the result only marginally. Using this approximation, the arm length ratio that maximizes (23) is $r = 1.33$ and the maximum GDI is 0.73. The plots in Fig. 4(c) show that for $l_{11}/d_{11} = 1.33$ both singular values are of similar size in a large section of the workspace. This means that the speed amplification in different directions are comparable in that section of the workspace. However, it should be remembered that it is normally just as important to have similar manipulator performance in the same direction, but in different positions. This means that the variation of $\sigma_1$ and $\sigma_2$ over the workspace should be minimized. For the studied manipulator, $\sigma_1$ is independent of the arm length ratio but instead of using (23) an alternative is to optimize the arm length ratio using a criterion that also penalizes the deviations of $\sigma_1$ from a constant value. The absolute sizes of $\sigma_1$ and $\sigma_2$ also need to be considered. The effects of low values are similar to using actuation with a high gear ratio, i.e. better resolution for the manipulated platform and better stiffness properties.

Workspace considerations are normally also important when determining the ratio of the arm lengths. For optimal size of the workspace the arms should be of similar length. This way the workspace extends close to the base column and the non-reachable disc in the center is minimized. There have been attempts to expand the optimization criteria with an added workspace dependent term [11]. Depending on the robot application, optimal arm lengths will vary and an optimization should include the relevant targets from those discussed above. As will be shown in the next subsection, for more complex manipulators in the same family, properties other than speed amplification and workspace will also be affected by the choice of arm lengths.

B. Finding the optimal Trilink Duorot variant

A natural way to achieve 3-DOF motion for the investigated family of manipulators is to use three identical kinematic chains as shown in the figures 1(e) and 1(f). However, it is also possible to use 2-DOF manipulators of the type shown in the figures 1(c) and 1(d) and either replace one lower arm link with an actuated telescopic link or mount a wrist on the manipulated platform. In both these cases it is advantageous to use a 2-DOF manipulator with minimal inherent position dependency on the platform rotation.

Although this approach of extending the DOFs of a manipulator is of less interest for the simple manipulators discussed here, this is a viable alternative for more complex spatial PKMs with a common axis of rotation, such as the SCARA-Tau manipulator [4]. The same mechanical design that minimizes the position dependence of the platform yaw angle of the 2-DOF manipulators studied here is immediately useful for some of these manipulators. Thus, this subsection focuses on determining the Trilink Duorot manipulator which gives the least platform variation along a radial line from the center of the base column, also considering the importance of the length ratio between the lower arm links and the upper arms. Variants where the distance $(d_{11} - d_{12})$ between $U_{11}$ and $U_{12}$ is larger than the length $r_{12}$ are not particularly useful since they limit the reach of the manipulator. Avoiding these parameter choices leaves four distinct variants worth investigating.

The figures 5(a), 5(b), 5(c) and 5(d) show four examples of Trilink Duorot manipulators, each plotted for three different values of $q_1$. The four manipulators have identical parameter values $r_{12} = r_{21} = 0.2m, \phi_{12} = -135$ degrees, $d_{11} = d_{21} = 1m$, and $l_{11} = l_{12} = l_{21} = 1.33m$. The only parameter that differs between the manipulators is $d_{12}$. Fig. 5(e) displays the yaw angles for the same four manipulators using solid lines and matching colors. Four additional plots using slightly different values of $d_{12}$ have been included using dashed lines. As can be seen in Fig. 5(e), there is a considerable difference in the range of yaw angle variation along the values of $d_{12}$. The reachable workspace is different for each of the variants. The minimum reachable radial position is limited by collision between the base column and the lower arm links and has been calculated assuming the base column radius is 0.1m and that the radii of all lower arm links are 0.02m.

Fig. 5(a) shows the Parallel Trilink Duorot, where the two lower arm links $L_{11}$ and $L_{12}$ are sections of a parallelogram. The yaw angle for this variant is solely related to the angle of the corresponding upper arm ($\phi = q_1 - \phi_{12} - \frac{\pi}{2}$). It is plotted with a solid black line in Fig. 5(e). In Fig. 5(b) a Convex Quadrilateral Trilink Duorot is displayed. The links $L_{11}$ and $L_{12}$ are sections of a convex quadrilateral shape between a parallelogram and a triangle. This variant is plotted with a solid red line in Fig. 5(e). The yaw angles for two other variants of this type with slightly different values of $d_{12}$ are plotted with dashed red lines in the same figure. Fig. 5(c) shows the Triangular Trilink Duorot. Here the platform is actuated by two lower arm links in a triangular configuration. The corresponding yaw angle is plotted with a solid blue line in Fig. 5(e). Fig. 5(d) shows the Intersecting Trilink Duorot where, when seen from above, the links $L_{11}$ and $L_{12}$ are intersecting. The corresponding yaw angle is plotted with a solid green line in Fig. 5(e). The yaw angles for two other variants of this type with slightly different values of $d_{12}$ are plotted in the same figure using dashed green lines.

According to the plots of the yaw angle in Fig. 5(e), a value of $d_{12}$ in the range 1.0 – 1.1 seems to give minimum variation of the yaw angle over the workspace. An optimal choice of $d_{12}$ is determined by minimizing the cost function

$$G(d_{12}) = \frac{h_{x_{\text{max}}}}{x_{\text{max}} - x_{\text{min}}} \sum_{x=1}^{x_{\text{max}}} (\phi(d_{12}, x) - \phi_m(d_{12}))^2,$$

where
The value of $\phi(d_{12}, x)$ is determined using (9) and $x_{\text{min}}$ and $x_{\text{max}}$ are the limits of the reachable workspace in the x-direction for the used parameter combination. Using the same parameters as in Fig. 5 and the step length $h_x = 0.001m$ the cost function is minimized for $d_{12} = 1.05m$. This is the same value which is used in the top plot with a dashed red line in Fig. 5(e).

As stated in (9), the angle $\phi_{12}$ gives a constant contribution to the yaw angle but does not affect the range of variation of the yaw angle over the workspace. The yaw angle is also independent of the parameter $r_{12}$, however, both the parameter $r_{12}$ and the length ratio between the lower arm links and the upper arms influence the range of yaw angle variation. In Fig. 6 the results from optimizing $d_{12}$ for minimum platform rotation are presented. The used parameters are in all cases $d_{11} = d_{21} = 1m$, $\phi_{12} = -135$ degrees and $r_{21} = 0.2m$. All three lower arm links have the same length $l_{21} = l_{12} = l_{11}$, but this length and the length of $r_{12}$ have been varied. An optimal value of $d_{12}$ has been determined for each parameter combination by minimizing the cost function (24). Figure 6(a) shows the optimal value of $d_{12}$ for combinations of six different lengths of the lower arm links ($l_{11} = 1.00m, 1.10m, 1.22m, 1.33m, 1.50m, 1.60m$) and three different platform sizes ($r_{12} = 0.2m, 0.3m, 0.4m$), while Fig. 6(b) shows the average absolute value of the deviation between the yaw angle and its mean value, utilizing the optimal value of $d_{12}$.

Figure 6(a) shows that for longer lower arm links (larger $l_{11}$) the optimal choice of $d_{12}$ approaches the value of $d_{11}$, which is one. Thus, for large arm length ratios the optimal choice of joint placements leads to configurations increasingly similar to the triangular configuration in Fig. 5(c). When the difference between the lengths of the upper arms and the lower arm links decreases (smaller $l_{11}$) the optimal difference between $d_{12}$ and $d_{11}$ increases, giving an optimal configuration more similar to the one in Fig. 5(b). For the studied parameter combinations the effect of larger values of $r_{12}$ is an increase in the optimal difference between $d_{12}$ and $d_{11}$. It is apparent from Fig. 6(b) that the smallest range of platform rotation is achieved when the upper arms and the lower arm links have the same length. The comparatively small influence of $r_{12}$ indicates that the effect of this parameter can largely be compensated for by different choices of $d_{12}$.
### Average Yaw Angle Deviation (Degrees)

<table>
<thead>
<tr>
<th>$l_{11}$ (m)</th>
<th>Average Yaw Angle Deviation (Degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>1.20</td>
<td>1.05</td>
</tr>
<tr>
<td>1.40</td>
<td>1.10</td>
</tr>
<tr>
<td>1.60</td>
<td>1.15</td>
</tr>
<tr>
<td>1.80</td>
<td>1.20</td>
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Fig. 6. Results from minimizing the platform yaw angle deviation with respect to $d_{12}$ for different values of the parameters $l_{11}$ and $r_{12}$. The three plots in Fig. (a) show the optimal value of $d_{12}$ as a function of $l_{11}$ for three different values of $r_{12}$. The plots in Fig. (b) show the average absolute value of the difference between the yaw angle and its mean value, calculated over the reachable workspace when using the optimal value of $d_{12}$.

### V. Conclusion and Future Work

A family of planar parallel manipulators has been analyzed and some novel members have been proposed. The common feature of the investigated manipulators is that the rotation axes of the actuated arms coincide. The main advantage of this feature is that the positional workspace is large in relation to the manipulator footprint. Both 2- and 3-DOF variants of the manipulators have been presented. Two of the analyzed 3-DOF manipulators utilize redundant actuators, making it possible to achieve infinite rotations of a platform in addition to the extensive positional workspace. By the proposed use of a passive telescopic link between two sections of the manipulated platform an over-constrained manipulated platform is avoided.

The investigated family of manipulators is suitable to use for kinematic analysis, as it is sufficient to investigate the properties of the manipulators along only one radial direction due to symmetry. The effect of different length ratios between the passive and actuated arms has been investigated using SVD of the Jacobian.

Mechanisms for planar manipulation with only two actuated arms have been analyzed. It has been shown that a convex quadrilateral link configuration gives a minimal position dependence of the platform rotation. It has also been demonstrated how this optimal link configuration depends on the platform size and the ratio between the lengths of the lower arm links and the upper arms.

The studied manipulators are of interest in their own right but many of the results, like the effect of different arm length ratios and the joint positions that minimize the dependence between position and orientation of the manipulated platform, are also immediately useful for more complex manipulators. Future work will include using these results to improve the performance of a 3-DOF spatial SCARA-Tau PKM.

### References